Appendix A

This appendix contains the benchmark problems used throughout this book. The problems are divided into categories based on the corresponding problem type. Information on each problem is reported when available, along with a relative reference.

UNCONSTRAINED OPTIMIZATION PROBLEMS

This section contains the unconstrained optimization problems used in the book at hand. Each problem is denoted with the general scheme, TP\textsubscript{UO-xx}, where xx is its number. Unless otherwise stated, we assume that \( x = (x_1, x_2, \ldots, x_n)^T \) is an \( n \)-dimensional vector.

\textbf{TP\textsubscript{UO-1} (Sphere)} (Storn & Price, 1997): This \( n \)-dimensional problem is defined as:

\[
f(x) = x^T x = \sum_{i=1}^{n} x_i^2
\]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0,0,\ldots,0)^T \).

\textbf{TP\textsubscript{UO-2} (Rosenbrock)} (Trelea, 2003): This \( n \)-dimensional problem is defined as,

\[
f(x) = \sum_{i=1}^{n-1} [(1-x_i)^2 + 100(x_{i+1} - x_i^2)^2]
\]

and it has a global minimum \( f^* = 0 \) at \( x^* = (1,1,\ldots,1)^T \). The 2-dimensional instance of the Rosenbrock function is also called \textit{Banana Valley} function.

\textbf{TP\textsubscript{UO-3} (Rastrigin)} (Storn & Price, 1997): This \( n \)-dimensional problem is defined as:
Appendix A

\[ f(x) = \sum_{i=1}^{n} [x_i^2 - 10 \cos(2\pi x_i) + 10] \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0,0,\ldots,0)^T \).

\( TP_{UO-4} \) (Griewank) (Storn & Price, 1997): This \( n \)-dimensional problem is defined as:

\[ f(x) = \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1 \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0,0,\ldots,0)^T \).

\( TP_{UO-5} \) (Schaffer’s F6) (Trelea, 2003): This 2-dimensional problem is defined as:

\[ f(x) = 0.5 + \left( \sin\left(\sqrt{x_1^2 + x_2^2}\right) \right)^2 - 0.5 \left( \frac{1 + 0.001(x_1^2 + x_2^2)}{2} \right)^2 \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0,0)^T \).

\( TP_{UO-6} \) (Ackley) (Storn & Price, 1997): This \( n \)-dimensional problem is defined as:

\[ f(x) = -20 \exp\left( -0.02 \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \right) - \exp\left( \frac{\sum_{i=1}^{n} \cos(2\pi i)}{n} \right) + 20 + \exp(1), \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0,0,\ldots,0)^T \).

\( TP_{UO-7} \) (Corana) (Storn & Price, 1997): This 4-dimensional problem is defined as:

\[ f(x) = \sum_{i=1}^{4} \begin{cases} 0.15(z_i - 0.05 \text{sign}(z_i))^2, & \text{if } |x_i - z_i| < 0.05, \\ d_i x_i^2, & \text{otherwise,} \end{cases} \]

where \( x_i \in [-1000,1000], d_1 = 1, d_2 = 1000, d_3 = 10, d_4 = 100, \) and,

\[ z_i = 0.2 \left( \frac{x_i}{0.2} \right) + 0.49999 \text{sign}(x_i). \]

All points with \( |x_i^*| < 0.05, i = 1, 2, 3, 4, \) are global minimizers with \( f^* = 0 \).

\( TP_{UO-8} \) (Lee & Yao, 2004): This \( n \)-dimensional problem is defined as:

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\[ f(x) = 0.1 \left\{ 1 + \sin^2(3\pi x_i) + \sum_{i=1}^{n-1} (x_i - 1)^2 + \sin^2(3\pi x_{i+1}) + (x_n - 1)^2 + \sin^2(2\pi x_n) \right\} + \sum_{i=1}^{n} u(x_i, 5, 100, 4), \]

where,
\[ u(z, a, k, m) = \begin{cases} 
  k(z - a)^m, & z > a, \\
  0, & -a \leq z \leq a, \\
  k(-z - a)^m, & z < -a,
\end{cases} \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (1, 1, \ldots, 1)^T \).

\textbf{TP\_UO-9} (Lee & Yao, 2004): This \( n \)-dimensional problem is defined as:
\[ f(x) = \frac{\pi}{n} \left\{ 10 + \sin^2(\pi x_i) + \sum_{i=1}^{n-1} (x_i - 1)^2 + \sin^2(\pi x_{i+1}) + (x_n - 1)^2 \right\} + \sum_{i=1}^{n} u(x_i, 10, 100, 4), \]

where \( u(x) \) is defined as in TP\_UO-8. It has a global minimum \( f^* = 0 \) at \( x^* = (1, 1, \ldots, 1)^T \).

\textbf{TP\_UO-10} (Branin) (Michalewicz, 1999): This 2-dimensional problem is defined as:
\[ f(x) = a_1(x_2 - a_2 x_1^2 + a_3 x_1 - a_4)^2 + a_5(1 - a_6)\cos(x_1) + a_7, \]

where, \( a_1 = 1, a_2 = 5.1/(4\pi^2), a_3 = 5/\pi, a_4 = 6, a_5 = 10, \) and \( a_6 = 1/(8\pi) \). It has three global minimizers, \( x_1^* = (-\pi, 12.275)^T, x_2^* = (\pi, 12.275)^T, \) and \( x_3^* = (9.42478, 2.475)^T, \) with \( f^* = 0.397887 \).

\textbf{TP\_UO-11} (Six-hump camel) (Michalewicz, 1999): This 2-dimensional problem is defined as:
\[ f(x) = (4 - 2.1x_1^2 + x_1^4/3)x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2, \]

and it has two global minimizers, \( x_1^* = (-0.0898, 0.7126)^T, x_2^* = (0.0898, -0.7126)^T, \) with \( f^* = -1.0316 \).

\textbf{TP\_UO-12} (Freudenberg-Roth) (More \textit{et al.}, 1981): This 2-dimensional problem is defined as:
\[ f(x) = (-13 + x_1 + ((5-x_2)x_2 - 2)x_2)^2 + (-29 + x_1 + ((x_2+1)x_2 - 14)x_2)^2, \]

and it has a global minimum \( f^* = 0 \) at \( x^* = (5, 4)^T \).

\textbf{TP\_UO-13} (Goldstein-Price) (Michalewicz, 1999): This 2-dimensional problem is defined as:
\[ f(x) = [1 + (x_1 + x_2 + 1)^2 (19-14x_1^3 + 3x_1^2 - 14x_2 + 6x_1x_2^2 + 3x_2^2)] \times [30 + (2x_1 + 3x_2)^2 (18-32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)], \]

and it has a global minimum \( f^* = 3 \) at \( x^* = (0, -1)^T \).
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TP \( UO-14 \) (Levy no. 3) (Levy et al., 1981): This 2-dimensional problem is defined as:

\[
f(x) = \sum_{i=1}^{5} i \cos((i-1)x_i + i) \times \sum_{j=1}^{5} j \cos((j+1)x_j + j),
\]

and it has 18 global minimizers with \( f^* = -176.542 \).

TP \( UO-15 \) (Levy no. 5) (Levy et al., 1981): This 2-dimensional problem is defined as:

\[
f(x) = \sum_{i=1}^{5} i \cos((i-1)x_i + i) \times \sum_{j=1}^{5} j \cos((j+1)x_j + j) + (x_i + 1.42513)^2 + (x_2 + 0.80032)^2,
\]

and it has a global minimum \( f^* = -176.1375 \) at \( x^* = (-1.3068, -1.4248)^T \).

TP \( UO-16 \) (Helical valley) (More et al., 1981): This 3-dimensional problem is defined as,

\[
f(x) = (10(x_3 - 100(x_1, x_2)))^2 + \left(10\left(\sqrt{x_1^2 + x_2^2} - 1\right)\right)^2 + x_3^2,
\]

where,

\[
\theta(x_1, x_2) = \begin{cases} 
\frac{1}{2\pi}\arctan(x_2 / x_1), & \text{if } x_1 > 0, \\
\frac{1}{2\pi}\arctan(x_2 / x_1) + 0.5, & \text{if } x_1 < 0,
\end{cases}
\]

and it has a global minimum \( f^* = 0 \) at \( x^* = (1, 0, 0)^T \).

TP \( UO-17 \) (Hyper-ellipsoid) (Storn & Price, 1997): This \( n \)-dimensional problem is defined as,

\[
f(x) = \sum_{i=1}^{n} i^2 x_i^2,
\]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0, 0, \ldots, 0)^T \).

TP \( UO-18 \) (Watson) (More et al., 1981): This \( n \)-dimensional problem is defined for \( 2 \leq n \leq 31 \) as,

\[
f(x) = \sum_{i=1}^{n} (f_i(x))^2,
\]

where,

\[
f_i(x) = \sum_{j=2}^{n} (j-1)x_j t_j^{i-2} - \left(\sum_{j=1}^{n} x_j t_j^{i-1}\right)^2 - 1,
\]
with \( t_i = i/29 \) for \( 2 \leq i \leq 29 \), while,

\[
f_{30}(x) = x_1 \quad \text{and} \quad f_{31}(x) = x_2 - x_1^2 - 1.
\]

For \( n = 6 \) it has a global minimum \( f^* = 0.00228767 \) at \( x^* = (0, 0, \ldots, 0)^T \).

**TP_{UO-19} (Levy no. 8) (Levy et al., 1981)**: This \( n \)-dimensional problem is defined as:

\[
f(x) = \sin^2(\pi y_1) + \sum_{i=1}^{n-1} (y_i - 1)^2[1 + 10 \sin^2(\pi y_{i+1})] + (y_n - 1)^2,
\]

where \( y_i = 1 + (x_i - 1)/4 \). For \( n = 3 \), it has a global minimum \( f^* = 0 \) at \( x^* = (1, 1, 1)^T \).

**TP_{UO-20} (Quadric) (Van den Bergh & Engelbrecht, 2002)**: This \( n \)-dimensional problem is defined as:

\[
f(x) = \left( \sum_{i=1}^n \sum_{j=1}^n x_i \right)^2,
\]

and it has a global minimum \( f^* = 0 \) at \( x^* = (0, 0, \ldots, 0)^T \).

**TP_{UO-21} (Egg holder) (Parsopoulos & Vrahatis, 2004)**: This 2-dimensional problem is defined as:

\[
f(x) = \cos^2(x_1) + \sin^2(x_2),
\]

and it has only global minimizers at the points \((k_1 \pi/2, k_2 \pi)^T\) with \( k_1 = \pm 1, \pm 2, \ldots \), and \( k_2 = 0, \pm 1, \pm 2, \ldots \). In the range \([-5, 5]^2\) it has 12 global minimizers.

**TP_{UO-22} (Beale) (More et al., 1981)**: This 2-dimensional problem is defined as:

\[
f(x) = [y_1 - x_1 (1-x_2)]^2 + [y_2 - x_1 (1-x_2)]^2 + [y_3 - x_1 (1-x_2)]^2,
\]

with \( y_1 = 1.5 \), \( y_2 = 2.25 \), and \( y_3 = 2.625 \). It has a global minimum \( f^* = 0 \) at \( x^* = (3, 0.5)^T \).

**NONLINEAR MAPPINGS**

This section contains the nonlinear mappings used for assessing the performance of the presented algorithms on problems of detecting periodic orbits. Each mapping is denoted with the general scheme, \( \text{TP}_{NM-xx} \), where \( xx \) is its number. Unless otherwise stated, we assume that \( x = (x_1, x_2, \ldots, x_n)^T \) is an \( n \)-dimensional vector.

**TP_{NM-1} (Hénon 2-dimensional) (Hénon, 1969)**: This is a 2-dimensional mapping defined as:
\[ \Phi(x) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} \Phi_1(x) = x_1 \cos a - (x_2 - x_1^2) \sin a, \\ \Phi_2(x) = x_1 \sin a - (x_2 - x_1^2) \cos a, \end{cases} \]

where \( a \in [0, \pi] \) is the rotation angle. The cases of \( \cos a = 0.8 \) and \( \cos a = 0.24 \) are very common.

**TP NM-2 (Standard)** (Rasband, 1990): This is a discontinuous 2-dimensional mapping defined as:

\[
\begin{align*}
\Phi_1(x) &= x_1 + x_2 - \frac{k}{2\pi} \sin(2\pi x_1) \mod \frac{1}{2}, \\
\Phi_2(x) &= x_2 - \frac{k}{2\pi} \sin(2\pi x_1) \mod \frac{1}{2},
\end{align*}
\]

where \( k = 0.9 \). The modulo function is defined as:

\[
y \mod \frac{1}{2} = \begin{cases} (y \mod 1) - 1, & \text{if } (y \mod 1) > 0.5, \\ (y \mod 1) + 1, & \text{if } (y \mod 1) < -0.5, \\ y \mod 1, & \text{otherwise.} \end{cases}
\]

**TP NM-3 (Gingerbreadman)** (Devaney, 1984): This is a nondifferentiable 2-dimensional mapping defined as:

\[
\begin{align*}
\Phi_1(x) &= 1 - x_2 + |x_1|, \\
\Phi_2(x) &= x_1,
\end{align*}
\]

and it has a unique periodic orbit of period \( p = 1 \).

**TP NM-4 (Predator-prey)** (Maynard Smith, 1968): This is a 2-dimensional mapping defined as:

\[
\begin{align*}
\Phi_1(x) &= ax_1(1-x_1) - x_1x_2, \\
\Phi_2(x) &= bx_1x_2,
\end{align*}
\]

where \( a = 3.6545 \) and \( b = 3.226 \) (Henry et al., 2000).

**TP NM-5 (Lorenz)** (Lorenz, 1963): This is a 3-dimensional mapping defined as:

\[
\begin{align*}
\Phi_1(x) &= \sigma(x_2 - x_1), \\
\Phi_2(x) &= rx_1 - x_2 - x_1x_3, \\
\Phi_3(x) &= x_1x_2 - bx_3,
\end{align*}
\]

where \( \sigma, r \) and \( b \) are the parameters of the system. Lorenz used the values \( \sigma = 10 \) and \( b = 8/3 \). Regarding \( r \), the value \( r = 28 \) is used (Henry et al., 2000).
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TP<sub>NM-6</sub> (Rössler) (Rössler, 1976): This is a 3-dimensional mapping defined as:

\[
\begin{align*}
\Phi_1(x) &= -(x_1 + x_2), \\
\Phi_2(x) &= x_1 + ax_2, \\
\Phi_3(x) &= b + x_3(x_1 - c),
\end{align*}
\]

where \(a, b,\) and \(c\) are parameters of the system. The values \(a = b = 0.2\) and \(c = 5.7\) are used (Henry et al., 2000).

TP<sub>NM-7</sub> (Hénon 4-dimensional) (Vrahatis, 1995): This 4-dimensional mapping is an extension of the 2-dimensional Hénon mapping defined previously as TP<sub>NM-1</sub> and it is defined as:

\[
\begin{pmatrix}
\Phi_1(x) \\
\Phi_2(x) \\
\Phi_3(x) \\
\Phi_4(x)
\end{pmatrix} = \begin{pmatrix}
R(a) & O \\
O & R(a)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 - x_1^2 + x_3^2 \\
x_3 \\
x_4 - 2x_1x_3
\end{pmatrix},
\]

where \(a\) is the rotation angle and \(R(a), O,\) are matrices defined as:

\[
R(a) = \begin{pmatrix}
\cos a & -\sin a \\
\sin a & \cos a
\end{pmatrix}, \quad O = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

The value \(a = \cos^{-1}(0.24)\) is a common choice. The mapping can be also generalized to a symplectic map with two frequencies, \(a_1\) and \(a_2\):

\[
\begin{pmatrix}
\Phi_1(x) \\
\Phi_2(x) \\
\Phi_3(x) \\
\Phi_4(x)
\end{pmatrix} = \begin{pmatrix}
R(a_1) & O \\
O & R(a_2)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 + x_1^2 - x_3^2 \\
x_3 \\
x_4 - 2x_1x_3
\end{pmatrix},
\]

TP<sub>NM-8</sub> (Kantz & Grassberger, 1988): This 6-dimensional mapping is the \(n = 3\) case of the standard maps studied by Kantz and Grassberger (1988) and it is defined as:

\[
\begin{align*}
x_1' &= x_1 + x_2' \\
x_2' &= x_2 + \frac{k}{2\pi} \sin(2\pi x_1) - \frac{b}{2\pi} \left[ \sin(2\pi(x_5 - x_1)) + \sin(2\pi(x_3 - x_1)) \right] \\
x_3' &= x_3 + x_4' \\
x_4' &= x_4 + \frac{k}{2\pi} \sin(2\pi x_3) - \frac{b}{2\pi} \left[ \sin(2\pi(x_1 - x_3)) + \sin(2\pi(x_5 - x_3)) \right] \quad \text{(mod 1)}.
\end{align*}
\]

\[
\begin{align*}
x_5' &= x_5 + x_6' \\
x_6' &= x_6 + \frac{k}{2\pi} \sin(2\pi x_4) - \frac{b}{2\pi} \left[ \sin(2\pi(x_3 - x_4)) + \sin(2\pi(x_1 - x_4)) \right]
\end{align*}
\]
All variables of the mapping are taken (mod 1), hence $x_i \in [0,1), i = 1, 2, \ldots, 6$. For $b = 0$, the mapping give three uncoupled standard maps, while for $b \neq 0$ the maps are coupled, influencing each other. In the present book, the values that were used in the experiments are $b = k = 1$.

**INVENTORY OPTIMIZATION PROBLEMS**

The continuous review inventory optimization problems used in the present book are based on the model of Chern et al. (2008):

$$TC(n,s_i,t_i) = \sum_{i=1}^{n} c_0 \exp(-r_i t_i)$$

$$+ \sum_{i=1}^{n} c_p \exp(-r_p t_i) \left( \int_{s_{i-1}}^{s_i} \beta(t_i - t) f(t) dt + \int_{s_i}^{s_{i+1}} \exp(\delta(t) - \delta(t_i)) f(t) dt \right)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{2} c_{ij} \int_{s_{i-1}}^{s_i} \exp(-r_j t_i) \int_{t_i}^{s_i} \exp(\delta(u) - \delta(t)) f(u) du dt$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{2} c_{ij} \int_{s_{i-1}}^{s_i} \left( \exp(-r_j t_i) - \exp(-r_j t_i) \right) \beta(t_i - t) f(t) dt$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{2} c_{ij} \int_{s_{i-1}}^{s_i} \exp(-r_j t_i) \left( 1 - \beta(t_i - t) \right) f(t) dt ,$$

with,

$$s_0 = 0, \quad s_{i-1} < t_i \leq s_i, \quad s_n = H.$$

The model can also admit constraints on the inventory size, resulting in the following global minimization problem:

$$\min_{n,t,s_i} \ TC(n,t,s_i) \quad such \ that \quad \int_{t_i}^{s_i} \exp(\delta(u) - \delta(t_i)) f(u) du \leq W,$$

$$s_0 = 0, \quad s_n = H, \quad s_{i-1} < t_i \leq s_i, \quad i = 1,2,\ldots,n.$$

All test problems are defined as the minimization problem above (either constrained or unconstrained) but for different parameter settings.

**TP_{INV-1}** (Skouri & Papachristos, 2002): This problem is based on a simplified version of the model, with parameters:

$$f(t) = 20 + 2t,$$

$$\beta(x) = \exp(-ax),$$
Appendix A

\[ r_1 = r_2 = 0, \]
\[ c_0 = 100, c_p = 0.2, c_{h_2} = 1.5, c_{i_2} = 0.5, c_{h_1} = 55, c_{h_1} = c_{h_2} = c_{i_1} = 0, \]
\[ \theta(t) = 0.01. \]

The problem was considered for three values of the parameter \( a \), namely \( a = 0.08, 0.05, \) and \( 0.02 \). In its constrained version, the maximum inventory size is set to the value, \( W = 90 \).

**TP_{INV,2}** (Chern et al., 2008): In this problem, shortages are completely backlogged. Its parameters are defined as follows:

\[ f(t) = 200 + 50t, \]
\[ \beta(x) = 1, \]
\[ H = 10, \]
\[ c_0 = 80, c_p = 9, c_{h_2} = 0.2, c_{h_2} = 0.4, c_{h_1} = 0.5, c_{h_2} = 0.4, \]
\[ r = 0.2, \]
\[ i_1 = 0.08, i_2 = 0.09, \]
\[ \theta(t) = 0.01. \]

In its constrained version, the maximum inventory size is set to the value, \( W = 300 \).

**TP_{INV,3}** (Chern et al., 2008): In this problem, shortages are completely backlogged. Its parameters are defined as follows:

\[ f(t) = 200 + 50t - 3r^2, \]
\[ \beta(x) = 1, \]
\[ H = 10, \]
\[ c_0 = 80, c_p = 15, c_{h_2} = 0.2, c_{h_2} = 0.4, c_{h_1} = 0.8, c_{h_2} = 0.6, \]
\[ r = 0.2, \]
\[ i_1 = 0.08, i_2 = 0.1, \]
\[ \theta(t) = 0.01. \]

In its constrained version, the maximum inventory size is set to the value, \( W = 300 \).

**GAME THEORY PROBLEMS**

The benchmark problems used for the detection of Nash equilibria were obtained through the state-of-the-art GAMBIT software suit (version 0.97.0.5) (Pavlidis et al., 2005). GAMBIT is available through the web address: http://gambit.sourceforge.net/. All problems have more than one Nash equilibria and their complete list can be obtained through the GAMBIT routine “PolEnumSolve”. For each test problem, its characteristics along with the corresponding defining GAMBIT file are reported.

**TP_{Ne,4}**: This normal-form game has 4 players, with 2 pure strategies for each one. It has 3 Nash equilibria and corresponds to the GAMBIT file “2x2x2x2.nfg”.
**TP\textsubscript{NE-2}**: This normal-form game has 4 players, with 2 pure strategies for each one. It has 5 mixed equilibria and corresponds to the GAMBIT file “g3.nfg”.

**TP\textsubscript{NE-3}**: This normal-form game has 5 players, with 2 pure strategies for each one. It has 5 Nash equilibria and corresponds to the GAMBIT file “2x2x2x2x2.nfg”.

**TP\textsubscript{NE-4}**: This normal-form game has 3 players, with 2 pure strategies for each one (McKelvey, 1991). The payoffs of the game are as follows:

<table>
<thead>
<tr>
<th>(s_{11})</th>
<th>(s_{21})</th>
<th>(s_{22})</th>
<th>(s_{23})</th>
<th>(s_{24})</th>
</tr>
</thead>
<tbody>
<tr>
<td>9, 8, 12</td>
<td>0, 0, 0</td>
<td>0, 0</td>
<td>9, 8, 2</td>
<td>0, 0, 0</td>
</tr>
</tbody>
</table>

It has 9 Nash equilibria (4 pure strategy, 3 mixed strategy, and 2 full support strategy equilibria). It corresponds to the GAMBIT file “2x2x2.nfg”.

**TP\textsubscript{NE-5}**: This coordination game has 3 players, with 3 strategies for each one. It has 13 equilibria and it corresponds to the GAMBIT file “coord333.nfg”.

**TP\textsubscript{NE-6}**: This game has 2 players, with 4 strategies for each one, and it has 15 Nash equilibria. Its payoff matrix is given as follows:

<table>
<thead>
<tr>
<th>(s_{11})</th>
<th>(s_{12})</th>
<th>(s_{13})</th>
<th>(s_{14})</th>
<th>(s_{21})</th>
<th>(s_{22})</th>
<th>(s_{23})</th>
<th>(s_{24})</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 2</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

The corresponding GAMBIT file is “coord4.nfg”.

**DATA SETS FROM BIOINFORMATICS**

The following data sets were considered for performance assessment of SA-PNNs by Georgiou \textit{et al.} (2006):

\textbf{E.coli data set} (Blake & Merz, 1998): The goal is to predict the cellular localization sites of \textit{E.coli} proteins. There are 8 cellular sites:

1. Cytoplasm (cp).
2. Inner membrane without signal sequence (im).
3. Periplasm (pp).
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4. Inner membrane with uncleavable signal sequence (imU).
5. Outer membrane (om).
6. Outer membrane lipoprotein (omL).
7. Inner membrane lipoprotein (imL).
8. Inner membrane with cleavable signal sequence (imS).

The attributes under consideration are the following:

4. Score of discriminant analysis of the amino acid content of outer membrane and periplasmic proteins (aac).
5. Score of the ALOM membrane spanning region prediction program (alm1).
6. Score of ALOM program after excluding putative cleavable signal regions from the sequence (alm2).

The size of the data set is 336 without any missing values and all its variables are continuous.

Yeast data set (Blake & Merz, 1998): The goal is to predict the cellular localization of yeast proteins. There are 10 sites:

1. CYT (cytosolic or cytoskeletal).
2. NUC (nuclear).
3. MIT (mitochondrial).
4. ME3 (membrane protein, no N-terminal signal).
5. ME2 (membrane protein, uncleaved signal).
6. ME1 (membrane protein, cleaved signal).
7. EXC (extracellular).
8. VAC (vacuolar).
9. POX (peroxisomal).
10. ERL (endoplasmic reticulum lumen).

The same attributes with the E.coli data set are considered, additionally including nuclear localization information. Thus, there are 8 continuous inputs of 1484 instances without missing values.

Breast Cancer data set (Prechelt, 1994): This data set was provided by the University of Wisconsin hospitals in 1992 and contains breast tumor records that can be categorized in two classes: benign and malignant. The input features are:

1. Uniformity of cell size and shape.
2. Bland chromatin.
4. Mitoses.
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There are 9 continuous inputs and 699 instances, without missing values.

**Pima Indians Diabetes data set** (Smith *et al.*, 1988): This data set was provided by the John Hopkins University in 1992 and concerns Pima Indians’ diabetes. The goal is to determine whether someone suffers from diabetes or not; hence, there are two classification classes. It has the following input features:

1. Diastolic blood pressure.
2. Triceps skin fold thickness.
3. Plasma glucose concentration in a glucose tolerance test.
4. Diabetes pedigree function.

Classification is performed on the patient’s exhibition of diabetes signs, based on criteria established by the World Health Organization. There are 8 inputs, all continuous without missing values, and 768 instances.

**MULTIOBJECTIVE OPTIMIZATION PROBLEMS**

The following widely used problems were used in experiments with multiobjective PSO variants:

**TP\textsubscript{MO-1}** (Knowles & Corne, 2000; Zitzler *et al.*, 2000): This problem has two objective functions, and it is defined as follows:

\[
    f_1(x) = \frac{1}{n} \sum_{i=1}^{n} x_i^2, \quad f_2(x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - 2)^2.
\]

It has a convex, uniform Pareto front.

**TP\textsubscript{MO-2}** (Knowles & Corne, 2000; Zitzler *et al.*, 2000): This problem has two objective functions, and it is defined as follows:

\[
    f_1(x) = x_1, \quad g(x) = 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i, \quad f_2(x) = g(x) \left( 1 - \sqrt{\frac{f_1(x)}{g(x)}} \right).
\]

It has a convex, non-uniform Pareto front.

**TP\textsubscript{MO-3}** (Knowles & Corne, 2000; Zitzler *et al.*, 2000): This problem has two objective functions, and it is defined as follows:

\[
    f_1(x) = x_1, \quad g(x) = 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i, \quad f_2(x) = g(x) \left( 1 - \left( \frac{f_1(x)}{g(x)} \right)^2 \right).
\]
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It has a concave Pareto front.

**TP\textsubscript{MO-4}** (Knowles & Corne, 2000; Zitzler et al., 2000): This problem has two objective functions, and it is defined as follows:

\[
\begin{align*}
    f_1(x) &= x_i, \\
    g(x) &= 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i, \\
    f_2(x) &= g(x) \left( 1 - \sqrt[4]{\frac{f_1(x)}{g(x)}} - \frac{f_1(x)}{g(x)} \sin(10\pi f_1(x)) \right).
\end{align*}
\]

It has neither a purely convex nor purely concave Pareto front.

**TP\textsubscript{MO-5}** (Knowles & Corne, 2000; Zitzler et al., 2000): This problem has two objective functions, and it is defined as follows:

\[
\begin{align*}
    f_1(x) &= x_i, \\
    g(x) &= 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i, \\
    f_2(x) &= g(x) \left( 1 - \sqrt[4]{\frac{f_1(x)}{g(x)}} - \frac{f_1(x)}{g(x)} \sin(10\pi f_1(x)) \right).
\end{align*}
\]

Its Pareto front consists of separated convex parts.

**CONstrained OPTIMIZATION PROBLEMS**

The following constrained optimization problems were used in experiments with constrained PSO approaches. The problems are divided in constrained benchmark and constrained engineering design problems.

**Constrained Benchmark Problems**

**TP\textsubscript{CO-1}** (Himmelblau, 1972): This 2-dimensional problem is defined as follows:

\[
f(x) = (x_1-2)^2 + (x_2-1)^2,
\]

subject to the constraints:

\[
\begin{align*}
    C_1(x): x_1 &= 2x_2 - 1, \\
    C_2(x): \frac{x_1^2}{4} + x_2^2 - 1 &\leq 0.
\end{align*}
\]

It has a solution with \(f^* = 1.3934651\).
TPco-2 (Floudas & Pardalos, 1990): This 2-dimensional problem is defined as follows:

\[ f(x) = (x_1 - 10)^3 + (x_2 - 20)^3, \]

subject to the constraints:

\[ C_1(x) = 100 - (x_1 - 5)^2 - (x_2 - 5)^2 \leq 0, \]
\[ C_2(x) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0, \]

with 13 ≤ x₁ ≤ 100 and 0 ≤ x₂ ≤ 100. It has a solution with \( f^* = -6961.81381 \).

TPco-3 (Hock & Schittkowski, 1981): This 7-dimensional problem is defined as follows:

\[ f(x) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7, \]

subject to the constraints:

\[ C_1(x) = -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \leq 0, \]
\[ C_2(x) = -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0, \]
\[ C_3(x) = -196 + 23x_1 + x_2^2 + 6x_3^4 - 8x_4 \leq 0, \]
\[ C_4(x) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0, \]

with -10 ≤ xᵢ ≤ 10, i = 1, 2,…, 7. It has a solution with \( f^* = 680.630057 \).

TPco-4 (Hock & Schittkowski, 1981): This 5-dimensional problem is defined as follows:

\[ f(x) = 5.3578547x_1^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141, \]

subject to the constraints:

\[ C_1(x) = 0 \leq 85.334407 + 0.0056858T_1 + T_2x_1x_4 - 0.0022053x_1x_5 \leq 92, \]
\[ C_2(x) = 90 \leq 80.51249 + 0.0071317x_1x_5 + 0.0029955x_1x_2 + 0.0021813x_2^2 \leq 110, \]
\[ C_3(x) = 20 \leq 9.300961 + 0.0047026x_1x_2 + 0.0012547x_1x_3 + 0.0019085x_3x_4 \leq 25, \]

with 78 ≤ x₁ ≤ 102, 33 ≤ x₂ ≤ 45, and 27 ≤ xᵢ ≤ 45, i = 3, 4, 5, where \( T_1 = x_2x_3 \) and \( T_2 = 0.0006262 \). It has a solution \( f^* = -30665.538 \).
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**TP\textsubscript{CO,5}** (Hock & Schittkowski, 1981): This problem is defined exactly as TP\textsubscript{CO,4} above, but with $T_1 = x_2x_3$ and $T_2 = 0.00026$.

**TP\textsubscript{CO,6}** (Michalewicz, 1999): This 5-dimensional problem is defined as follows:

$$f(x, y) = -10.5x_1 - 7.5x_2 - 3.5x_3 - 2.5x_4 - 1.5x_5 - 10y - 0.5\sum_{i=1}^{5} x_i^3,$$

subject to the constraints:

$$C_1(x): \quad 6x_1 + 3x_2 + 3x_3 + 2x_4 + x_2 - 6.5 \leq 0,$$

$$C_2(x): \quad 10x_1 + 10x_3 + y \leq 20,$$

with $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, 5$, and $y \geq 0$. It has a solution $f^* = -213.0$.

### Constrained Engineering Design Problems

**TP\textsubscript{ED,1} (Tension/compression spring)** (Arora, 1989): This problem concerns the minimization of weight for the tension/compression spring illustrated in Fig. 1, subject to constraints on minimum deflection, shear stress, surge frequency, diameter, and design variables. The design variables are the wire parameter, $d$, the mean coil diameter, $D$, and the number of active coils, $N$.

If $x = (d, D, N)^T$, the problem is defined as follows:

$$f(x) = (N + 2) D^2,$$

subject to the constraints:

$$C_1(x): \quad 1 - \frac{D^3 N}{71785d^4} \leq 0,$$

$$C_2(x): \quad \frac{4D^2 - DD}{12566(Dd^3 - d^4)} + \frac{1}{5108d^2} - 1 \leq 0,$$

$$C_3(x): \quad 1 - \frac{140.45d}{D^2 N} \leq 0,$$

$$C_4(x): \quad \frac{D + d}{1.5} - 1 \leq 0,$$

with $0.05 \leq d \leq 2.0$, $0.25 \leq D \leq 1.3$, and $2.0 \leq N \leq 15.0$. 
TP_{ED-2} (*Welded beam*) (Rao, 1996): This problem concerns the minimization of the cost of the welded beam illustrated in Fig. 2, subject to constraints on shear stress, \( \tau \); bending stress in the beam, \( \sigma \); buckling load on the bar, \( P_c \); end deflection of the bead, \( \delta \); and side constraints. There are 4 design variables, \( h \), \( l \), \( t \), and \( b \), henceforth denoted as \( x_1 \), \( x_2 \), \( x_3 \), and \( x_4 \), respectively.

If \( x = (x_1, x_2, x_3, x_4)^T \), the problem is defined as follows:

\[
f(x) = 1.10471x_1^2x_2 + 0.04811x_3x_4(14.0+x_2),
\]

subject to the constraints:

\[
\begin{align*}
C_1(x) & : \quad \tau(x) - \tau_{\text{max}} \leq 0, \\
C_2(x) & : \quad \sigma(x) - \sigma_{\text{max}} \leq 0, \\
C_3(x) & : \quad x_1 - x_4 \leq 0, \\
C_4(x) & : \quad 0.10471x_1^2 + 0.04811x_3x_4(14.0+x_2) - 5.0 \leq 0, \\
C_5(x) & : \quad 0.125 - x_1 \leq 0, \\
C_6(x) & : \quad \delta(x) - \delta_{\text{max}} \leq 0, \\
C_7(x) & : \quad P - P_c(x) \leq 0,
\end{align*}
\]

where:

\[
\begin{align*}
\tau(x) & = \sqrt{\tau'^2 + 2\tau'\tau''\frac{x_2}{2R} + \tau''^2}, \\
\tau' & = \frac{P}{\sqrt{2}x_1x_2}, \quad \tau'' = \frac{MR}{J}, \quad M = P\left(\frac{L + x_2}{2}\right), \\
R & = \sqrt{\frac{x_3^2}{4} + \left(\frac{x_1 + x_3}{2}\right)^2}, \quad J = 2\left[\sqrt{2}x_1x_2\left(\frac{x_2}{12} + \left(\frac{x_1 + x_3}{2}\right)^2\right)\right].
\end{align*}
\]
Figure 2. The welded beam problem

\[
\sigma(x) = \frac{6PL}{x_4x_5^2}, \quad \delta(x) = \frac{4PL}{Ex_4x_5}, \quad P(x) = \frac{4.013E}{L^2} \left( \frac{x_1^2x_4^6}{36} \right) \left( 1 - \frac{x_3}{2L} \sqrt{\frac{E}{4G}} \right),
\]

with \( P = 6000 \text{ lb}; L = 14 \text{ in}; E = 30 \times 10^6 \text{ psi}; G = 12 \times 10^6 \text{ psi}; \tau_{\text{max}} = 13600 \text{ psi}; \sigma_{\text{max}} = 30000 \text{ psi}; \delta_{\text{max}} = 0.25 \text{ in}; \) and \( 0.1 \leq x_i, x_4 \leq 2.0, 0.1 \leq x_2, x_3 \leq 10.0. \)

**TP_{ED-3} (Gear train)** (Sandgen, 1990): This problem concerns the minimization of the cost of the gear ratio of the gear train illustrated in Fig. 3. The gear ratio, \( gr \), is defined as follows:

\[
gr = \frac{n_Bn_D}{n_An_F},
\]

where \( n_j \) denotes the number of teeth of the \( j \)-th gearwheel, \( j = A, B, D, F \). The design variables, \( n_A, n_B, n_D, \) and \( n_F \), will be henceforth denoted as \( x_1, x_2, x_3, \) and \( x_4, \) respectively.

If \( x = (x_1, x_2, x_3, x_4)^T \), the problem is defined as follows:

\[
f(x) = \left( \frac{1}{6.931} - \frac{x_3x_4}{x_1x_2} \right)^2,
\]

subject to the constraints: \( 12 \leq x_i \leq 60, i = 1, 2, 3, 4. \)

**TP_{ED-4} (Pressure vessel)** (Sandgen, 1990): This problem concerns the minimization of the cost of the pressure vessel illustrated in Fig. 4. The design variables are shell thickness, \( T_s \); thickness of head, \( T_h \); inner radius, \( R \); and the length, \( L \), which will be henceforth denoted as \( x_1, x_2, x_3, \) and \( x_4, \) respectively. \( T_s \) and \( T_h \) are integer multiples of 0.0625, representing the available thicknesses of rolled steel plates.
If \( x = (x_1, x_2, x_3, x_4)^T \), the problem is defined as follows:

\[
f(x) = 0.6224x_1x_3x_4 + 1.7781x_2^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3,
\]

subject to the constraints:

\[
\begin{align*}
C_1(x) & : -x_1 + 0.0193x_3 \leq 0, \\
C_2(x) & : -x_2 + 0.00954x_3 \leq 0, \\
C_3(x) & : -\pi x_3^2x_4 - (4/3)\pi x_3^3 + 1296000 \leq 0, \\
C_4(x) & : x_4 - 240 \leq 0,
\end{align*}
\]

with \( 1 \leq x_1, x_2 \leq 99, 10 \leq x_3, x_4 \leq 0 \).

**MINIMAX PROBLEMS**

Minimax problems are intimately related to constrained optimization problems. As reported in Chapter Eleven of the book at hand, a minimax problem can be given in the explicit form:

\[
\min_x \max_{i=1}^m \{ f_i(x) \},
\]

where \( i = 1, 2, \ldots, m \), denotes the number of functions. Also, a constrained problem of the form:

\[
\min_x f(x) \text{ subject to the constraints } C_i(x) \geq 0, \quad i = 2, 3, \ldots, m,
\]

can be given in an equivalent implicit minimax form:

---

*Figure 3. The gear train problem*
Appendix A

\[
\min\max_{x \in \mathbb{R}^m} \{ f_i(x) \},
\]

\[
f_i(x) = f(x),
\]

\[
f_j(x) = f(x) - a_j C_j(x),
\]

\[a_j > 0, \quad j = 2,3,\ldots,m.\]

The following widely used minimax test problems were used for experimentation with PSO, given in either of the aforementioned forms:

**TP\textsubscript{MX-1} (Xu, 2001):** This 2-dimensional problem is defined directly as a minimax problem, as follows:

\[
\min\max_{x \in \mathbb{R}^3} \{ f_i(x) \},
\]

\[
f_1(x) = x_1^2 + x_2^4,
\]

\[
f_2(x) = (2-x_1)^2 + (2-x_2)^2,
\]

\[
f_3(x) = 2\exp(-x_1^2 + x_2).
\]

**TP\textsubscript{MX-2} (Xu, 2001):** This 2-dimensional problem is similar to the previous one, defined as follows:

\[
\min\max_{x \in \mathbb{R}^3} \{ f_i(x) \},
\]

\[
f_1(x) = x_1^4 + x_2^2,
\]

\[
f_2(x) = (2-x_1)^2 + (2-x_2)^2,
\]

\[
f_3(x) = 2\exp(-x_1^2 + x_2).
\]

**TP\textsubscript{MX-3} (Rosen-Suzuki problem) (Xu, 2001):** This 4-dimensional problem is defined in implicit form, as follows:

\[
f_1(x) = f(x),
\]

\[
f_j(x) = f(x) - a_j C_j(x),
\]

\[a_j > 0, \quad j = 2,3,\ldots,m.\]

Figure 4. The pressure vessel problem
min max \{f_i(x)\},

f_i(x) = \begin{cases} x_1^2 + x_2^2 + 2x_3^2 + 4x_4^2 - 5x_5 - 5x_6 - 21x_7 + 7x_8, \\ f_i(x) = \begin{cases} x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5 + x_6 + x_7 + 8, \\ f_i(x) = -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_5 + x_6 + 10, \\ f_i(x) = -x_1^2 - x_2^2 - x_3^2 - 2x_4 + x_5 + x_6 + 5, \\ \end{cases} \end{cases}

C_i(x) = \begin{cases} x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_5 + x_6 + 10, \\ C_i(x) = -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_5 + x_6 + 10, \\ C_i(x) = -x_1^2 - x_2^2 - x_3^2 - 2x_4 + x_5 + x_6 + 5, \\ \end{cases}

TP_{MX,4} \text{(Xu, 2001): This 7-dimensional problem is defined in implicit form, as follows:}

\begin{align*}
\min & \max \{f_i(x)\}, \\
f_i(x) &= f(x) = (x_1-10)^2 + 5(x_1-12)^2 + 3(x_2-11)^2 + x_3^4 + 10x_4^6 + 7x_5^2 + x_6^4 - 4x_7 - 10x_8 - 8x_9, \\
f_i(x) &= f(x) - a_i C_i(x), \quad i = 2, 3, 4, 5,
\end{align*}

with,

C_2(x) = -2x_1^2 - 3x_2^4 - x_3 - 4x_4^2 - 5x_5 + 127, \\
C_3(x) = -7x_1 - 3x_2 - 10x_3^2 - x_4 + x_5 + 282, \\
C_4(x) = -23x_1 - x_2^2 - 6x_3^2 + 8x_4 + 196, \\
C_5(x) = -4x_1^2 - x_2^2 + 3x_1 x_2 - 2x_3^2 - 5x_4 + 11x_5

TP_{MX,5} \text{(Schwefel, 1995): This 2-dimensional problem is defined in explicit minimax form, as follows:}

\begin{align*}
\min & \max \{f_i(x)\}, \\
of_i(x) &= |x_1 + 2x_2 - 7|, \\
of_i(x) &= |2x_1 + x_2 - 5|.
\end{align*}

TP_{MX,6} \text{(Schwefel, 1995): This 10-dimensional problem is defined in explicit minimax form, as follows:}

\begin{align*}
\min & \max \{f_i(x)\}, \\
f_i(x) &= \begin{cases} |x_1|, \quad i = 1, 2, \ldots, 10. \\
\end{cases}
\end{align*}

FURTHER LINKS FOR OPTIMIZATION TEST PROBLEMS

Researchers can find a plethora of optimization test problems in the following web sources:

http://titan.princeton.edu/TestProblems/
http://plato.la.asu.edu/bench.html
Appendix A

http://www-optima.amp.i.kyoto-u.ac.jp/member/student/hedar/Hedar_files/TestGO.htm
http://www.netlib.org/toms/667
http://www.netlib.org/uncon/data/
http://www.mat.univie.ac.at/~neum/glopt/test_results.html
http://www.mat.univie.ac.at/~neum/glopt/my_problems.html
http://infohost.nmt.edu/~borchers/optlibs.html
http://www.nada.kth.se/~viggo/wwwcompendium/wwwcompendium.html
http://www.ti3.tu-harburg.de/deutsch/optimierung.html
http://riot.ieor.berkeley.edu/riot/index.html

as well as in the software for generation of classes of test functions of Gaviano et al. (2003).

REFERENCES


Appendix A


