Neural/Fuzzy Computing Based on Lattice Theory

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INTRODUCTION

Computational Intelligence (CI) consists of an evolving collection of methodologies often inspired from nature (Bonissone, Chen, Goebel & Khedkar, 1999, Fogel, 1999, Pedrycz, 1998). Two popular methodologies of CI include neural networks and fuzzy systems.

Lately, a unification was proposed in CI, at a “data level”, based on lattice theory (Kaburlasos, 2006). More specifically, it was shown that several types of data including vectors of (fuzzy) numbers, (fuzzy) sets, 1D/2D (real) functions, graphs/trees, (strings of) symbols, etc. are partially (lattice)-ordered. In conclusion, a unified cross-fertilization was proposed for knowledge representation and modeling based on lattice theory with emphasis on clustering, classification, and regression applications (Kaburlasos, 2006).

Of particular interest in practice is the totally-ordered lattice $(\mathbb{R}, \leq)$ of real numbers, which has emerged historically from the conventional measurement process of successive comparisons. It is known that $(\mathbb{R}, \leq)$ gives rise to a hierarchy of lattices including the lattice $(\mathbb{F}, \leq)$ of fuzzy interval numbers, or FINs for short (Papadakis & Kaburlasos, 2007).

This article shows extensions of two popular neural networks, i.e., fuzzy ARTMAP (Carpenter, Grossberg, Markuzon, Reynolds & Rosen 1992) and self-organizing map (Kohonen, 1995), as well as an extension of conventional fuzzy inference systems (Mamdani & Assilian, 1975), based on FINs. Advantages of the aforementioned extensions include both a capacity to rigorously deal with nonnumeric input data and a capacity to introduce tunable nonlinearities. Rule induction is yet another advantage.

BACKGROUND

Lattice theory has been compiled by Birkhoff (Birkhoff, 1967). This section summarizes selected results regarding a Cartesian product lattice $(L, \leq) = (L_1, \leq) \times \ldots \times (L_N, \leq)$ of constituent lattices $(L_i, \leq)$, $i=1, \ldots, N$.

Given an isomorphic function $\theta: (L_i, \leq) \to (L'_i, \leq)$ in a constituent lattice $(L_i, \leq)$, $i=1, \ldots, N$, where $(L_i, \leq)$ then an isomorphic function $\theta: (L, \leq) \to (L'_i, \leq)$ is given by $\theta(x_1, \ldots, x_N) = (\theta(x_1), \ldots, \theta(x_N))$.

A positive valuation function $v_i: (L_i, \leq) \to \mathbb{R}$ in a constituent lattice $(L_i, \leq)$, $i=1, \ldots, N$ then a positive valuation $v: (L, \leq) \to \mathbb{R}$ is given by $v(x_1, \ldots, x_N) = v_1(x_1) + \ldots + v_N(x_N)$.

It is well-known that a positive valuation $v_i: (L_i, \leq) \to \mathbb{R}$ in a lattice $(L, \leq)$ implies a metric function $d_i: L \times L \to \mathbb{R}^+$ given by $d(a, b) = v(a \wedge b) - v(a \vee b)$.

Minkowski metrics $d_p: (L_i, \leq) \times \ldots \times (L_N, \leq) \to \mathbb{R}$ are given by

$$d_p(x, y) = \left[ d_1^p(x_1, y_1) + \ldots + d_N^p(x_N, y_N) \right]^{1/p},$$

where

$x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N), p \in \mathbb{R}$.

An interval $[a, b]$ in a lattice $(L, \leq)$ is defined as the set $[a, b] = \{ x \in L: a \leq x \leq b, a, b \in L \}$. Let $\tau(L)$ denote the set of intervals in a lattice $(L, \leq)$. It turns out that $(\tau(L), \leq)$ is a lattice, ordered by set inclusion.

Definition 1. The size $Z_p: \tau(L) \to \mathbb{R}^+_0$ of a lattice $(L, \leq)$ interval $[a, b] \in \tau(L)$, with respect to a positive valuation $v: (L, \leq) \to \mathbb{R}$, is defined as

$Z_p([a, b]) = d_p(a, b)$.

NEURAL/FUZZY COMPUTING BASED ON LATTICE THEORY

This section delineates modified extensions to a hierarchy of lattices stemming from the totally ordered lattice $(\mathbb{R}, \leq)$ of real numbers. Then, it details the relevance of...
novel mathematical tools. Next, based on the previous mathematical tools, this section presents extensions of ART/SOM/FIS. Finally, it discusses comparative advantages.

**Modified Extensions in a Hierarchy of Lattices**

Consider the product lattice \((\Delta, \leq) = (\mathbb{R} \times \mathbb{R}, \leq)\) of generalized intervals. A generalized interval (element in \(\Delta\)) will be denoted by \([a, b]\) and will be called positive (negative) for \(a \leq b\) (\(a > b\)). The set of positive (negative) generalized intervals will be denoted by \(\Delta^+ (\Delta^-)\). We remark that the set of positive generalized intervals is isomorphic to the set of conventional intervals in the set \(\mathbb{R}\) of real numbers.

A decreasing function \(\theta: \mathbb{R} \to \mathbb{R}\) is an isomorphic function \(\theta: (\mathbb{R}, \leq) \to (\mathbb{R}, \leq)\); furthermore, a strictly increasing function \(v: \mathbb{R} \to \mathbb{R}\) is a positive valuation in lattice \((\Delta, \leq)\). There follows a metric function \(d(\Delta) = d\): \(\Delta \times \Delta \to \mathbb{R}^+_0\) given by \(d((a, b]) = v[\theta h(a)] + v(b)\) is a positive valuation in lattice \((\Delta, \leq)\). The following integral exists, a metric function \(d_\alpha: G \times G \to \mathbb{R}^+_0\) is given by

\[
d_\alpha(G_1, G_2) = \int_0^1 d_\alpha(G(h), G_2(h)) dh.
\]

Our interest here focuses on the sublattice \((F, \leq)\) of lattice \((G, \leq)\), namely sublattice of fuzzy interval numbers (FINs). A FIN is defined rigorously as follows.

**Definition 4.** A fuzzy interval number (FIN) \(F\) is a GIN such that either (1) both \(F(h) \in \Delta\), \(h \in [0, 1]\) (positive FIN) or (2) there is a positive FIN \(F\) such that \(F(h) = -F(h)\), for all \(h \in [0, 1]\) (negative FIN).

**Relevance of Novel Mathematical Tools**

A fundamental mathematical result in fuzzy set theory is the “resolution identity theorem”, which states that a fuzzy set can, equivalently, be represented either by its membership function or by its \(\alpha\)-cuts (Zadeh, 1975). The aforementioned theorem has been given little attention in practice to date. However, some authors have capitalized on it by designing effective as well as efficient fuzzy inference systems (FIS) involving fuzzy numbers whose \(\alpha\)-cuts are conventional closed intervals (Uehara & Fujise, 1993, Uehara & Hirota, 1998).

This work builds on the abovementioned mathematical result as follows. In the first place, we drop the possibilistic interpretation of a membership function. Then, we consider the corresponding “\(\alpha\)-cuts representation”.

We remark that the cardinality of set \(G\) equals \(N_1 = (2^{N_1})^{N_1} = 2^{N_2} = N_2 > N_1\), where \(N_1\) is the cardinality of the set \(\mathbb{R}\) of real numbers.