Chapter 7
Classified Discrete-Time Markov Chains

ABSTRACT

The main focus of this chapter is on the formalization of classified DTMCs. The chapter begins by presenting the formalization of some foundational notions of classified states, which are categorized based on reachability, periodicity, or absorbing features. Then, these results along with the formal definition of a DTMC, presented in the previous chapter, are used to formalize classified Markov chains, such as aperiodic and irreducible DTMCs. Based on these concepts, some long-term properties are verified for the purpose of formally checking the correctness of the functions of Markovian systems or analyzing the performance of Markov chain models.
7.1 FORMALIZATION OF CLASSIFIED STATES

The foremost concept of states classification is the first passage time \( \tau_j \), which is sometimes referred to as the first hitting time. It is defined as the minimum time required to reach a state \( j \) from the initial state \( i \):

\[
\tau_j = \min \{ t > 0: X_t = j \}
\]  

(7.1)

The first passage time can be defined in HOL4 as:

Definition 7.1
\[ \forall X \times j. \text{FPT} X \times j = \text{MINSET} \{ t | 0 < t \land (X \cdot t \cdot x = j) \} \]

where \( X \) is a random process and \( x \) is a sample in the probability space associated with the random variable \( X_t \). Note that the first passage time is also a random variable.

The conditional distribution of \( \tau_j \) defined as the probability of the events starting from state \( i \) and visiting state \( j \) at time \( n \) is expressed as \( f_{ij}^{(n)} = \Pr \{ \tau_j = n | X_0 = i \} \).

This definition can be formalized in HOL4 as (Liu, 2013):

Definition 7.2
\[ \forall X \times p \times i \times j \times n. \quad f X \times p \times i \times j \times n = \text{prob} p (\{ x | \text{FPT} X \times j = n \} | \{ x | X \cdot 0 \cdot x = i \}) \]

Another important notion, denoted as \( f_{jj} \), is the probability of the events starting from state \( i \) and visiting state \( j \) at all times \( n \), is expressed as \( f_{jj} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \). It can be expressed in HOL4 as \((\lambda n. f X \times p \times i \times j \times n) \) sums \( f_{ij} \). Thus \( f_{jj} \) provides the probability of events starting from state \( j \) and eventually returning back to \( j \). If \( f_{jj} = 1 \), then the mean return time of state \( j \) is defined as \( \mu_j = \sum_{n=1}^{\infty} n f_{jj}^{(n)} \). The existence of this infinite summation can be specified as \( \text{summable} \ (\lambda n. n * f X \times p \times j \times j \times n) \) in HOL.

A state \( j \) in a DTMC \( \{ X_t \}_{t \geq 0} \) is called transient if \( f_{jj} < 1 \), and persistent if \( f_{jj} = 1 \). If the mean return time \( \mu_j \) of a persistent state \( j \) is finite, then \( j \) is said to be persistent non null state (or positive persistent state). Similarly, if \( \mu_j \) is infinite, then \( j \) is termed as persistent null state.

The greatest common divisor (gcd) of a set is a frequently used mathematical concept in defining classified states. We formalize the gcd of a set as follows: