Superior Cantor Sets and Superior Devil Staircases

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ABSTRACT

Mandelbrot, in 1975, coined the term fractal and included Cantor set as a classical example of fractals. The Cantor set has wide applications in real world problems from strange attractors of nonlinear dynamical systems to the distribution of galaxies in the universe (Schroder, 1990). In this article, we obtain superior Cantor sets and present them graphically by superior devil’s staircases. Further, based on their method of generation, we put them into two categories.

Keywords: Cantor Set, Classical Fractal, Devil’s Staircase, Fractal, Superior Cantor Set, Superior Devil’s Staircase, Superior Iterates

INTRODUCTION

The Cantor set, originally constructed for purely abstract fascination, lately turned into near perfect models for a host of phenomena in the real world—from strange attractors of nonlinear dynamical systems to the distribution of galaxies in the universe (Schroder, 1990). In modern terminology, the Cantor set is a classical example of a perfect subset of the closed interval [0, 1] that has the same cardinality as the real line but whose Lebesgue measure is zero.

Since its advent, the Cantor set finds celebrated place in mathematical analysis and its applications. For a fundamental work relevant to the study of the Cantor set and Devil’s staircase, one may refer to Peitgen, Jürgen, and Saupe (2004). However, for an excellent study of the Cantor set in n-dimensional spaces, and beautiful applications, one may refer to Schroeder (1990). Devaney (1992) has given a good account of the Cantor set in chaotic dynamical system, while Dovgoshey, Martio, Ryazanov, and Vuorinen (2006) gave a systematic survey of properties of the Cantor ternary map. Indeed, the Cantor set has widely been studied from different aspects (see recent work Pickover et al., 1998; Solomyak, 1997; Thiele, 2005).

Devil’s staircase is the graphical presentation of the Cantor set. See its various aspects, for instance, in Aubry (1983), Chen (1987), Horiguchi et al. (1984); Qu et al. (1997), Rani et al. (2008), Troll (1991), Zyczkowski (1999). It has its applications in different branches of science and engineering. In order to appreciate some of its applications in Chemistry, one may refer to (Gutfraind et al., 1990; Kuznik et al., 1993; Schroder, 1990).
Recently, Rani, and Kumar (2004a, 2004b) introduced superior iterations in the study of fractals and chaos and described this powerful tool to improve various geometric objects effectively in a series of papers (Kumar et al., 2005; Negi et al., 2008; Rani et al., 2005, 2008, 2009). In this article, inspired by the success of superior iterations in the study of fractals and its applications, we generate superior Cantor sets. Further, we present them graphically via superior Devil’s staircases. The work embodied in this article is organized in the following manner. In Section 2, we give some preliminaries to be used in the sequel. The intent of Section 3 is to obtain superior Cantor sets and generate superior devil’s staircases by running computer programs. Conclusion of the work is given in Section 4, while the algorithm used to generate a superior Devil’s staircase is given in the last section.

PRELIMINARIES

The Devil’s staircase, a graphical presentation of the middle one-third Cantor set, is shown in Figure 1. The flat parts are the images of all of the middle thirds, and these are all connected itself by the images of the Cantor set. This construction has infinitely many irregular steps, and therefore, named as Devil’s staircase (Peitgen et al., 2004). The dimension of the Cantor middle one-third set $\approx 0.6309$ and its distribution is as follows:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{\sum_{n=0}^{\infty} 2^n}{3^n}$$

Let $A$ be a subset of real or complex numbers and $f: A \rightarrow A$. For $x_0 \in A$, construct a sequence $\{x_n\}$ in $A$ in the following manner:

$$x_n = \beta_n f(x_{n-1}) + (1 - \beta_n) x_{n-1},$$

where $0 < \beta_n \leq 1$ and $\{\beta_n\}$ is convergent away from zero. This definition of the sequence $\{x_n\}$, essentially due to Mann (1953), is called superior iterates in discrete dynamics (see Kumar et al., 2005; Negi et al., 2008; Rani et al., 2004a, 2004b, 2005, 2008, 2009).

The sequence $\{x_n\}$ constructed above is usually denoted by $SO(f, x_0, \beta_n)$ and called the superior orbit. Notice that the superior orbit $SO(f, x_0, \beta_n)$ with $\beta_n = 1$ is the well known Picard orbit.

SUPERIOR CANTOR SETS AND SUPERIOR DEVIL’S STAIRCASES

Inspired by superior iterations, we compute the superior Cantor sets and generate the corresponding superior Devil’s staircases. We classify them in two categories.

Figure 1. Scaling figure $(r_1, r_2, r_3, :1/3, 1/3, 1/3)$

(a) Cantor middle one third set

(b) Devil’s staircase
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