Algorithms and Computations in BL-Algebras

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ABSTRACT

In this paper, the authors study the notion of n-fold implicative and fuzzy n-fold implicative filters, which correspond to various sets of provable formulae, in BL-algebras. Several characterizations of fuzzy n-fold implicative filters are given and the authors analyse some relationships with various n-fold and fuzzy n-fold filters in BL-algebras, while constructing some algorithms for studying the foldness of filters in BL-algebras. The authors describe the relation between fuzzy n-fold implicative filters and fuzzy (n+1)-fold implicative filters and establish the extension property for fuzzy n-fold implicative filters in BL-algebras.

Keywords: BL-Algebra, Filter, Fuzzy Filter, Fuzzy n-Fold Implicative Filter, n-Fold Implicative Filter

INTRODUCTION

Basic logic (BL for short) was introduced by Hájek (1988) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on a real segment [0,1] and the implication by its corresponding adjoint. He also introduced BL-algebras as the algebraic counterpart of this logic. Since then, many researchers have investigated various properties of these algebras. For the general development of BL-algebras, filters theory plays an important role.

From the logical point of view, various filters correspond to various sets of provable formulae. The BL-algebras are classes of algebras that also satisfy some BCI/BCK/MV-algebras properties. Hence the problem of getting similar results for these structures is natural. Recently, L. Liu and K. Li (2005a, 2005b) introduced the notion of fuzzy filters in BL-algebras. In Lele (2009), we have studied the notion of n-fold, fuzzy n-fold positive implicative filters in BL-algebras and established many important properties. All the above interesting results motivate us to further investigate the fuzzy foldness of other types of filters in BL-algebras and analyse the relation diagram between them. In this work, we generalize the notion of filters and fuzzy filters in BL-algebras (Xueling, Zhan, & Dudek, 2009; Kondo & Dudek, 2008; Cignoli & Torrens, 2005; Agliano & Montagna, 2003).

The paper recalls some basic notions on BL-algebras and list some results involved in the sequel. Next, we analyse some properties of n-fold implicative filters in BL-algebras. Then, we study of fuzzy n-fold implicative filters in BL-algebras. We first give some properties of fuzzy n-fold implicative filters in BL-algebras,

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then we prove that every fuzzy $n$-fold implicative filter is a fuzzy filter, but the converse is not true. Thanks to a level set of a fuzzy set in a BL-algebra, we give a characterization of fuzzy $n$-fold implicative filters in BL-algebra. Furthermore, we describe the relation between fuzzy $n$-fold implicative filters and fuzzy $(n+1)$-fold implicative filters and we establish the extension property for fuzzy $n$-fold implicative filters in BL-algebras. The results of are used in the rest of the paper to analyse some relationships with various $n$-fold and fuzzy $n$-fold filters in BL-algebras. We obtain that every fuzzy $n$-fold positive implicative filter is also a fuzzy $n$-fold implicative filter. One of the difficult task while studying BL-algebras, is to construct some examples. Therefore, in the appendix of the paper, we give some algorithms for the computations in BL-algebras. All the results in this paper are the natural generalization of the notion of filters and fuzzy filters (namely deductive and fuzzy deductive systems) in BL-algebras (Zhan & Yang, 2008; Haveshki, Saied, & Esrami, 2006; Hajek, 1988; Dumitru & Dana, 2003; Jun & Miko, 2004; Turunen, 1999; Turunen, 1999; Turunen & Sessa, 2001).

**Preliminaries**

A BL-algebra is a structure $(X, \land, \lor, *, \rightarrow, 0, 1)$ in which $X$ is a non empty set with four binary operations $\land, \lor, *, \rightarrow$ and two constants 0 and 1 satisfying the following axioms:

BL-1 $(X, \land, \lor, 0, 1)$ is a bounded lattice;
BL-2 $(X, *, 1)$ is an abelian monoid, i.e., $*$ is commutative and associative with $x \cdot 1 = x$;
BL-3 $x \land y \leq z$ if $x \leq y \rightarrow z$ (Residuation);
BL-4 $x \land y = x \lor (x \rightarrow y)$ (Divisibility);
BL-5 $(x \rightarrow y) \lor (y \rightarrow x) = 1$ (Prelinearity).

A BL-algebra $X$ is called an MV-algebra if $-(-x) = x$, or equivalently $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, where $-x = x \rightarrow 0$.

**Example 2.1** Let $I$ be a unit interval $[0,1]$ and let $*$ and $\rightarrow$ two binary operations defined by

\[
\begin{align*}
x \cdot y &= \min(x, y) \quad \text{(with } x \rightarrow y = 1) \quad \text{if } x \leq y, \\
x \rightarrow y &= y \quad \text{otherwise.}
\end{align*}
\]

Therefore, $(I, \leq, \min, \max, *, \rightarrow, 0, 1)$ is a BL-algebra.

The following properties also hold in any BL-algebra (Haveshki, Saied, & Esrami, 2006; Hajek, 1988; Dumitru & Dana, 2003; Jun & Miko, 2004; Turunen, 1999; Turunen, 1999; Turunen & Sessa, 2001):

1. $x \leq y$ iff $x \rightarrow y = 1$;
2. $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow z$;
3. $x \land x \land y \leq y$;
4. $(x \rightarrow y) \land (y \rightarrow z) \leq x \rightarrow z$;
5. $x \lor y = ((x \rightarrow y) \land (y \rightarrow x)) \rightarrow x$;
6. $x \lor (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
7. $x \rightarrow (y \rightarrow z) \leq \rightarrow (z \rightarrow x)$;
8. $y \rightarrow (z \rightarrow y) \leq (z \rightarrow x)$;
9. If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$;
10. $y \leq (y \rightarrow x)$ and $y \leq (x \rightarrow y)$;
11. $x \leq y \rightarrow (x \rightarrow y)$;
12. $x \land (x \rightarrow y) \leq y$;
13. $1 \rightarrow x = x$; $x \rightarrow x = 1$; $x \rightarrow 1 = 1$; $x \rightarrow 0 \rightarrow x$.

We briefly review some concepts of fuzzy logic and we refer the reader to Jun, Shim, & Lele, 2002; Lele & Moutari, 2007; Lele & Moutari, 2008; Lele, Wu, & Mamadou, 2002; DiLascio, Fischetti, & Gisolfi, 2008; DiLascio, Fischetti, & Gisolfi, 2006; Neggers & Kim, 2000; Liu, Liu, & Xu, 2006; Liu & Li, 2005; and Zadeh, 1965, for more details.

**Definition 2.1** A fuzzy subset of a BL-algebra $X$ is a function $A : X \rightarrow [0,1]$. We denoted $\ldots (x \rightarrow (x \rightarrow (x \rightarrow y))) \ldots$ by $x^n \rightarrow y$ where $x$ occurs $n$ times for all $x, y \in X$. Using the fact that in any BL-algebra
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