Algorithms and Computations in BL-Algebras

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ABSTRACT

In this paper, the authors study the notion of n-fold implicative and fuzzy n-fold implicative filters, which correspond to various sets of provable formulae, in BL-algebras. Several characterizations of fuzzy n-fold implicative filters are given and the authors analyse some relationships with various n-fold and fuzzy n-fold filters in BL-algebras, while constructing some algorithms for studying the foldness of filters in BL-algebras. The authors describe the relation between fuzzy n-fold implicative filters and fuzzy (n+1)-fold implicative filters and establish the extension property for fuzzy n-fold implicative filters in BL-algebras.

Keywords: BL-Algebra, Filter, Fuzzy Filter, Fuzzy n-Fold Implicative Filter, n-Fold Implicative Filter

INTRODUCTION

Basic logic (BL for short) was introduced by Hájek (1988) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on a real segment [0, 1] and the implication by its corresponding adjoint. He also introduced BL-algebras as the algebraic counterpart of this logic. Since then, many researchers have investigated various properties of these algebras. For the general development of BL-algebras, filters theory plays an important role.

From the logical point of view, various filters correspond to various sets of provable formulae. The BL-algebras are classes of algebras that also satisfy some BCI/BCK/MV-algebras properties. Hence the problem of getting similar results for these structures is natural. Recently, L. Liu and K. Li (2005a, 2005b) introduced the notion of fuzzy filters in BL-algebras. In Lele (2009), we have studied the notion of n-fold, fuzzy n-fold positive implicative filters in BL-algebras and established many important properties. All the above interesting results motivate us to further investigate the fuzzy foldness of other types of filters in BL-algebras and analyse the relation diagram between them.

In this work, we generalize the notion of filters and fuzzy filters in BL-algebras (Xueling, Zhan, & Dudek, 2009; Kondo & Dudek, 2008; Cignoli & Torrens, 2005; Agliano & Montagna, 2003). The paper recalls some basic notions on BL-algebras and list some results involved in the sequel. Next, we analyse some properties of n-fold implicative filters in BL-algebras. Then, we study of fuzzy n-fold implicative filters in BL-algebras. We first give some properties of fuzzy n-fold implicative filters in BL-algebras,

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then we prove that every fuzzy \( n \)-fold implicative filter is a fuzzy filter, but the converse is not true. Thanks to a level set of a fuzzy set in a BL-algebra, we give a characterization of fuzzy \( n \)-fold implicative filters in BL-algebras. Furthermore, we describe the relation between fuzzy \( n \)-fold implicative filters and fuzzy \((n+1)\)-fold implicative filters and we establish the extension property for fuzzy \( n \)-fold implicative filters in BL-algebras. The results of are used in the rest of the paper to analyse some relationships with various \( n \)-fold and fuzzy \( n \)-fold filters in BL-algebras. We obtain that every fuzzy \( n \)-fold positive implicative filter is also a fuzzy \( n \)-fold implicative filter. One of the difficult task while studying BL-algebras, is to construct some examples. Therefore, in the appendix of the paper, we give some algorithms for the computations in BL-algebras. All the results in this paper are the natural generalization of the notion of filters and fuzzy filters (namely deductive and fuzzy deductive systems) in BL-algebras (Zhan & Yang, 2008; Haveshki, Saied, & Esfandi, 2006; Hajek, 1988; Dumitru & Dana, 2003; Jun & Miko, 2004; Turunen, 1999; Turunen, 1999; Turunen & Sessa, 2001):

\[
\begin{align*}
\text{(1)} & \quad x \leq y \iff x \to y = 1; \\
\text{(2)} & \quad x \to (y \to z) = (x \ast y) \to z; \\
\text{(3)} & \quad x \ast y \leq x \land y; \\
\text{(4)} & \quad (x \to y) \ast (y \to z) \leq x \to z; \\
\text{(5)} & \quad x \lor y = ((x \to y) \to y) \land ((y \to x) \to x); \\
\text{(6)} & \quad x \to (y \to z) = y \to (x \to z); \\
\text{(7)} & \quad x \to y \leq (y \to z) \to (x \to z); \\
\text{(8)} & \quad y \to x \leq (z \to y) \to (z \to x); \\
\text{(9)} & \quad \text{If } x \leq y, \text{ then } y \to z \leq x \to z \text{ and } z \to x \leq y; \\
\text{(10)} & \quad y \leq (y \to x) \to x; \\
\text{(11)} & \quad x \leq y \to (x \ast y); \\
\text{(12)} & \quad x \ast (x \to y) \leq y; \\
\text{(13)} & \quad 1 \to x = x; \quad x \to x = 1; \quad x \to 1 = 1; \\
& \quad x \leq y \to x. 
\end{align*}
\]

Therefore, \((I, \leq, \min, \max, \ast, \to, 0, 1)\) is a BL-algebra.

The following properties also hold in any BL-algebra (Haveshki, Saied, & Esfandi, 2006; Hajek, 1988; Dumitru & Dana, 2003; Jun & Miko, 2004; Turunen, 1999; Turunen, 1999; Turunen & Sessa, 2001):

\[
\begin{align*}
\text{(1)} & \quad x \leq y \iff x \to y = 1; \\
\text{(2)} & \quad x \to (y \to z) = (x \ast y) \to z; \\
\text{(3)} & \quad x \ast y \leq x \land y; \\
\text{(4)} & \quad (x \to y) \ast (y \to z) \leq x \to z; \\
\text{(5)} & \quad x \lor y = ((x \to y) \to y) \land ((y \to x) \to x); \\
\text{(6)} & \quad x \to (y \to z) = y \to (x \to z); \\
\text{(7)} & \quad x \to y \leq (y \to z) \to (x \to z); \\
\text{(8)} & \quad y \to x \leq (z \to y) \to (z \to x); \\
\text{(9)} & \quad \text{If } x \leq y, \text{ then } y \to z \leq x \to z \text{ and } z \to x \leq y; \\
\text{(10)} & \quad y \leq (y \to x) \to x; \\
\text{(11)} & \quad x \leq y \to (x \ast y); \\
\text{(12)} & \quad x \ast (x \to y) \leq y; \\
\text{(13)} & \quad 1 \to x = x; \quad x \to x = 1; \quad x \to 1 = 1; \\
& \quad x \leq y \to x. 
\end{align*}
\]

**Preliminaries**

A BL-algebra is a structure \((X, \land, \lor, \ast, \to, 0, 1)\) in which \(X\) is a non empty set with four binary operations \(\land, \lor, \ast, \to\) and two constants 0 and 1 satisfying the following axioms:

- **BL-1** \((X, \land, \lor, 0, 1)\) is a bounded lattice;
- **BL-2** \((X, \ast, 1)\) is an abelian monoid, i. e., \(\ast\) is commutative and associative with \(x \ast 1 = x\);
- **BL-3** \(x \ast y \leq z \iff x \leq y \to z\) (Residuation);
- **BL-4** \(x \land y = x \ast (x \to y)\) (Divisibility);
- **BL-5** \(x \to y \lor (y \to x) = 1\) (Prelinearity).

A BL-algebra \(X\) is called an MV-algebra if \(\neg(\neg x) = x\), or equivalently \((x \to y) \to y = (y \to x) \to x\), where \(\neg x = x \to 0\).

**Example 2.1** Let \(I\) be a unit interval \([0,1]\) and let \(\ast\) and \(\to\) two binary operations defined by

\[
\begin{align*}
x \ast y &= \min(x, y) \quad \text{(with } x \leq y, \\
x \to y &= y \quad \text{otherwise.} 
\end{align*}
\]

Therefore, \((I, \leq, \min, \max, \ast, \to, 0, 1)\) is a BL-algebra.

We briefly review some concepts of fuzzy logic and we refer the reader to Jun, Shim, & Lele, 2002; Lele & Moutari, 2007; Lele & Moutari, 2008; Lele, Wu, & Mamadou, 2002; DiLascio, Fischetti, & Gisolfi, 2008; DiLascio, Fischetti, & Gisolfi, 2006; Neggers & Kim, 2000; Liu, Liu, & Xu, 2006; Liu & Li, 2005; and Zadeh, 1965, for more details.

**Definition 2.1** A fuzzy subset of a BL-algebra \(X\) is a function

\[
A : X \to [0,1].
\]

We denoted \(((x \to (x \to (x \to y))))\) by \(x^n \to y\) where \(x\) occurs \(n\) times for all \(x, y \in X\). Using the fact that in any BL-algebra