Existence of Positive Solutions of Nonlinear Second-Order M-Point Boundary Value Problem

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ABSTRACT

Under suitable conditions on \( f(t, u) \), the nonlinear second-order m-point boundary value problem

\[
\begin{align*}
    u''(t) + f(t, u(t)) &= 0, \quad 0 < t < 1 \\
    u(0) &= 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)
\end{align*}
\]

has at least one positive solution. In this paper, the authors examine the positive solutions of nonlinear second-order m-point boundary value problem.

Keywords: Fixed Point Theorem in Cone, Multi-Point, Nonlinear, Positive Solution, Second-Order

INTRODUCTION

The study of multi-point boundary value problems for linear second-order ordinary value problems was initiated by Llyin and Moisser (1987). Then Gupta studied three-point boundary value problems for nonlinear ordinary differential equations. We refer the reader to the reference (Gupta et al., 1994; Feng et al., 1997; Ma, 2001; Sun, 2005) for some recent results of nonlinear multi-point boundary value problems.

In this paper, we investigate the existence of positive solutions to nonlinear second-order multi-point boundary value problems:

\[
\begin{align*}
    u''(t) + f(t, u(t)) &= 0, \quad 0 < t < 1 \\
    u(0) &= 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)
\end{align*}
\]

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where $a_i \geq 0$ for $i = 1, 2, \ldots, m - 3$

and $a_{m-2} > 0$, $\xi_i$ satisfy $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ and

$$\sum_{i=1}^{m-2} a_i u(\xi_i) < 1.$$ 

We also make the following assumptions.

$$\text{(A}_k\text{)} f \in C\left([0,1] \times [0,\infty);[0,\infty]\right)$$

$$(A_{k'})$$ There are real numbers $H_1$, $H_2$ satisfying $f(s,u) \leq M_1 H_1$ on $[0,1] \times [0, H_1]$ and $f(s,u) \geq M_2 H_2$ on $[\xi_{m-2}, 1] \times [\Gamma H_2, \infty)$, where $M_1 = 2 \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right)$ and $M_2 = 2 \left(1 - \sum_{i=1}^{m-2} a_i \xi_i \right)$ or $\frac{1}{(1 - \xi_{m-2})^2 \sum_{i=1}^{m-2} a_i \xi_i}$.

$$(A_{k''})$$ There are real numbers $H_3$, $H_4$ satisfying $f(s,u) \geq M_2 H_3$ on $[\xi_{m-2}, 1] \times [0, \Gamma H_3]$ and $f(s,u) \leq M_1 H_4$ on $[0,1] \times [H_4, \infty)$.

The proof of the main result in the article is based on an application of the following well-known Guo-Krasnoselskii fixed-point theorem (Krasnoselkii, 1964; Guo & Lakshmikantham, 1998).

**The Preliminary Lemmas**

**Lemma 2.1** (Gupta et al., 1994). Let $a_i \geq 0$ for $i = 1, 2, \ldots, m - 2$ and $\sum_{i=1}^{m-2} a_i \xi_i \neq 1$; then for $y \in C[0,1]$, the boundary value problem:

$$u''(t) + g(t) = 0, \quad 0 < t < 1, \quad (2.1)$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \quad (2.2)$$

has a unique solution

$$u(t) = \frac{t}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \int_0^t (1 - s)g(s)ds - \int_0^t g(s)ds - \frac{t}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \int_0^\xi (\xi_i - s)g(s)ds$$

Let $C^+[0,1]$ be the set of nonnegative function in $C[0,1]$.

**Lemma 2.2** (R. Ma, 2001). Let $a_i \geq 0$ for $i = 1, 2, \ldots, m - 2$ and $\sum_{i=1}^{m-2} a_i \xi_i < 1$, then for $y \in C^+[0,1]$, the unique solution $u(t)$ of (2.1), (2.2) is nonnegative and satisfies

$$\min_{t\in[t_0,1]} u(t) \geq \Gamma \|u\|$$

where

$$\Gamma = \min \left\{ \frac{a_{m-2}(1 - \xi_{m-2})}{1 - a_{m-2}^2 \xi_{m-2}}, a_{m-2} \xi_{m-2}, \xi_1 \right\}.$$  

**Main Results**

In this section we show the existence of positive solution for the boundary value problem (1.1), (1.2).

**Theorem 3.1.** Suppose $(A_1),(A_2)$ hold, then (1.1), (1.2) has at least one positive solution.
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