Chapter  7

Existence of Positive Solutions for Generalized p–Laplacian BVPs

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ABSTRACT

Using Kransnoskii’s fixed point theorem, the authors obtain the existence of multiple solutions of the following boundary value problem

(BVP)

\[ \begin{align*}
(E) & \left( \varphi_p \left( u^{(n-1)}(t) \right) \right)' + f(t, u(t), ..., u^{(n-2)}(t)) = 0, \quad t \in (0,1), \\
(BC) & \begin{cases}
  u^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\
  u^{(n-2)}(0) - B_0 \left( u^{(n-1)}(\xi) \right) = 0, \\
  u^{(n-2)}(1) + B_1 \left( u^{(n-1)}(\eta) \right) = 0,
\end{cases}
\end{align*} \]

where \(0 < \xi < \eta < 1\) are given. The authors examine and discuss these solutions.

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1. INTRODUCTION

In this paper, we concern with the existence of multiple solutions for higher order boundary value problem

\[
\begin{cases}
(E) & \left| f \left( t, u(t), \ldots, u^{(n-3)}(t) \right) \right| + \phi_p(u^{(n-1)}(t)) = 0, \quad t \in [0,1), \\
(BC) & u^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\
& u^{(n-i)}(1) + B_i(u^{(n-i)}(t)) = 0,
\end{cases}
\]

where \( n \geq 3 \) is a positive integer, \( 0 < \xi < \eta < 1 \) are given and \( \varphi_p(s) \) is the p-Laplacian operator, that is, \( \varphi_p(s) = |s|^{p-2}s \) for \( p > 1 \). Clearly, \( \varphi_p \) is invertible with inverse \( \varphi_q = \varphi_p^{-1}(s) \). Here \( \frac{1}{p} + \frac{1}{q} = 1 \).

In recent years, the existence of positive solutions for nonlinear boundary value problems with p-Laplacian operator received wide attention. As we know, two point boundary value problems are used to describe a number of physical, biological and chemical phenomena. Recently, some authors have obtained some existence results of positive solutions of multi-points boundary value problems for second order ordinary differential equations (Wang & Ge, 2007; Yu, Wong, Yeh, & Lin, 2007; Zhao, Wang, & Ge, 2007; Zhou, & Su, 2007). In this paper, we establish the existence of positive solutions of general multi-points boundary value problem (BVP) and related results (Bai, Gui, & Ge, 2004; Guo & Lakshmikantham, 1988; Guo, Lakshmikantham, & Liu, 1996; He & Ge, 2004; Lian & Wong, 2000; Liu, 2002; Ma, 1999; Ma & Cataneda, 2001; Sun, Ge, & Zhao, 2007; Wang, 1997).

In order to abbreviate our discussion, throughout this paper, we assume

\( (H_1) \ f \in C \left( [0,1] \times [0, +\infty)^{n-1}, [0, +\infty) \right); \)

\( (H_2) \ B_0(s), B_1(s) \text{ are both nondecreasing continuous and odd functions defined on } (-\infty, +\infty) \text{ and at least one of them satisfies the condition that there exists } b \geq 0 \text{ such that } 0 \leq B_i(s) \leq bs \text{ for all } s \geq 0, \ i = 1, 2. \)

2. PRELIMINARIES AND LEMMAS

Let

\[ B = \left\{ u \in C^{(n-2)}[0,1]: u^{(i)} = 0, \ 0 \leq i \leq n-3 \right\}. \]

Then, \( B \) is a Banach space with norm \( \|u\| = \max_{t \in [0,1]} |u^{(n-2)}(t)| \). And let

\[ K = \left\{ u \in B: u^{(n-2)}(t) \geq 0 \text{ is a concave function, } t \in [0,1] \right\}. \]

Obviously, \( K \) is a cone in \( B \).

In order to discuss our results, we need the following some lemmas:

Lemma 2.0

Assume that \( E \) is a Banach space and \( P \subset E \) is a cone in \( E \); \( \Omega_1, \Omega_2 \) are open subsets of \( E \), and \( 0 \in \overline{\Omega_1} \subset \Omega_2 \). Furthermore, let \( F: P \cap \left( \overline{\Omega_2} \setminus \Omega_1 \right) \rightarrow P \) be a completely continuous operator satisfying one of the following conditions:

(i) \( \|Fx\| \leq \|x\|, \quad \forall x \in P \cap \partial \Omega_1; \) \( \|Fx\| \geq \|x\|, \quad \forall x \in P \cap \partial \Omega_2; \)

(ii) \( \|Fx\| \leq \|x\|, \quad \forall x \in P \cap \partial \Omega_2; \) \( \|Fx\| \geq \|x\|, \quad \forall x \in P \cap \partial \Omega_1. \)

Then \( F \) has a fixed point in \( P \cap \left( \overline{\Omega_2} \setminus \Omega_1 \right) \).