On Some New Fractional Type Heinz Inequalities

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ABSTRACT

The authors shall discuss Heinz inequalities involving Riemann-Liouville fractional integrals for certain unitarily invariant norms. By using the convexity of function and fractional Hermit-Hadamard integral inequality, some refinements of Heinz inequalities are derived.

Keywords: Convexity of Function, Fractional Integrals, Heinz Inequalities, Riemann-Liouville, Unitarily Invariant Norm

1. INTRODUCTION

It is well known that Heinz means is one of special means which can interpolate between the geometric and the arithmetic mean. The Heinz means are defined as:

\[ H_v(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}, \quad v \in [0,1] \] (1)

for positive numbers a and b. For \( v = 0.1 \), the inequality (1) is equal to the arithmetic mean, and for \( v = 0.5 \), that inequality (1) is equal to the geometric mean.

Moreover, one can show that:

\[ \sqrt{ab} \leq H_v(a,b) \leq \frac{a + b}{2} \]

The general matrix version of the above asserts were proved in Bhatia and Davis (1993). Let A, B and X are three operators on a complex separable Hilbert space such that A and B are positive. For every unitarily invariant norm \( \| \cdot \| \), the function:

\[ f(v) = \| A^v XB^{1-v} + A^{1-v} XB^v \|, \quad v \in [0,1] \] (2)
is convex on the interval \([0, 1]\), arrives its maximum at \(v = 0\) and \(v = 1\), attains its minimum at \(v = \frac{1}{2}\) and satisfies 
\[ f(v) = f(1 - v) \quad \text{for} \quad v \in [0, 1]. \]
Thus, we have the following standard operator version Heinz inequalities:

\[ 2 ||| A^2 X B^2 ||| \leq f \left( \frac{1}{2} \right) \leq ||| AX + XB ||| \]  

It is easy to see that the old Heinz operator version inequality in Heinz (1951) is just the special case of the inequalities (3) whose norm is the operator bound norm.

For more refinements and applications on the inequalities (3), the reader can refer to Bhatia (2007), Hiai and Kosaki (1999), Hiai and Kosaki (2003), Kittaneh (2010), and Feng (2012) and references therein. Among these obtained results, the following known Hermit-Hadamard integral inequality for a convex function \(f\):

\[ f(a) + f(b) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq f \left( \frac{a + b}{2} \right) \]  

which plays a very important role in the research. In fact, the inequality (4) was firstly discovered by Hermite in 1881 in the journal Mathesis (Mitrinovic & Lackovic, 1985). However, this beautiful result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result (Pečarić, Proschan, & Tong, 1992). For more recent results which generalize, improve, and extend the inequality (4), one can see (Abramovich, Barić, & Pečarić, 2008; Cal, Carcamo, & Escauriaza, 2009; Demir, Avci, & Set, 2010; Dragomir, 2011; Dragomir, 2012; Sarikaya & Aktan, 2011; Xiao, Zhang, & Wu, 2011; Bessenyei, 2010; Tseng, Hwang, & Hsu, 2012; Niculescu, 2012) and references therein.

Very recently, Sarikaya et al. (2012) derive a new refinement of the inequality (4) involving Riemann-Liouville fractional integrals as follows: Lemma 1.1. (Fractional version Hermit-Hadamard inequality) Let \(f : [a, b] \to \mathbb{R}\) be a positive function with \(0 \leq a < b\) and \(f \in L[a, b]\). If \(f\) is a convex function on \([a, b]\), then the following inequality for fractional integrals hold:

\[ f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(a + 1)}{2(b - a)^a} \left[ RL^a \! _f (b) + RL^b \! _f (a) \right] \leq \frac{f(a) + f(b)}{2} \]  

where the symbol \(RL^a \! _f + f\) and \(RL^b \! _f - f\) denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order \(\alpha \in \mathbb{R}^+\) are defined by:

\[
(\text{RL}^a \! _f + f)(x) = \frac{1}{\Gamma(a)} \int_a^x (x - t)^{a-1} f(t) \, dt, \quad (0 \leq a < x \leq b)
\]

and:

\[
(\text{RL}^b \! _f - f)(x) = \frac{1}{\Gamma(a)} \int_a^b (t - x)^{a-1} f(t) \, dt, \quad (0 \leq a \leq x < b)
\]

respectively. Here \(\Gamma(\cdot)\) is the Gamma function.

Fractional calculus have recently proved to be a powerful tool for the study of dynamical properties of many interesting systems in physics, chemistry, and engineering. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more recent development on fractional calculus, one can see the monographs (Baleanu, Machado, & Luo, 2012; Diethelm, 2010; Kilbas, Srivastava, & Trujillo, 2006; Lakshmikantham, Leela, & Vasundhara
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