Design of a Functional H Filter for Linear Singular Systems With an Additional Unknown Input, Bounded Disturbance, and Variable Time Delay

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ABSTRACT

This paper deals with a new method for the design of a functional filter for linear singular systems with bounded disturbance and variable time delay in the state and the known input vectors. Moreover, a disturbance unknown inputs is considered. The main objective of such design is the estimation of functional state vector, and a part of the disturbance unknown inputs for fault detection. The proposed design is based on Lyapunov-Krasovskii stability theory and, in order to minimize the disturbance effect, the designed filter satisfies a H_∞ criterion. Numerical simulation is used to illustrate the proposed design and the results were successful.

KEYWORDS

Bounded Disturbance, Functional Filter, H_∞ Criterion, Singular Systems, Unknown Inputs, Variable Time Delay

1. INTRODUCTION

Singular systems have been widely considered in the last decade (Dai, 1989; Darouach, 1995; Ezzine, 2010; Khadhraoui, 2014). Since standard state space are not applicable for such system description, singulars models have been developed for such purpose. Therefore, to generate the state vector we have to design an observer. When an unknown disturbance input is acting on such system, we have to estimate such input especially when we aim to handle fault detection or system control. Even there are some works that dealt with the design of an observer for linear singular systems in presence of additional unknown input (Darouach, 1996; Ezzine, 2011; Khadhraoui, 2014; Johnson, 1975), no attempt has been made for the estimation of such input.

Moreover, the use of sensors and actuators introduce time delay on the system input and state. Therefore, the designed observer should take into account the delays effect. Despite that the simplest approach would consist in replacing them by some finite-dimensional approximations (Fattouh, 1999; Darouach, 2005; Ezzine, 2011; Khadhraoui, 2014), constant delay is not the general method

DOI: 10.4018/IJSDA.2020100102

This article, originally published under IGI Global's copyright on October 1, 2020 will proceed with publication as an Open Access article starting on January 25, 2021 in the gold Open Access journal, International Journal of System Dynamics Applications (converted to gold Open Access January 1, 2021), and will be distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>) which permits unrestricted use, distribution, and production in any medium, provided the author of the original work and original publication source are properly credited.

of modeling delay. Also, in last decade, many researches (Koeing, 2006; Souley, 2006; Khadhraoui, 2016) have been interested in the design of an observer for system with Variable Time Delay.

Filters for singular systems with additional unknown inputs and constant time delay have been widely considered in the last decade and several design methods have been proposed (Ezzine, 2011; Khadhraoui, 2016). However, only few researches (Darouach, 2005; Khadhraoui, 2016) have been proposed when the delay is variable.

Motivated by these facts, we propose a functional H_∞ filter for singular unknown inputs system with the presence of variable time delay in state and known input vector. Moreover, it doesn't require a prior knowledge on the disturbance except its bounds.

Since the dynamics of the estimation error contains the derivative of the bounded disturbance a transformation has been made so that the structure of this dynamics becomes a singular system. This feature enables to find the optimum gain of the filter by resolution of Linear Matrix Inequalities (LMI) so that we estimate a functional of the state vector and a part of the considered unknown input.

The outlines of the paper are as follows. Section 2 presents the difference between current work and recent published work. The third section gives the problem to be solved is formulated. Section 4 presents the contribution of the paper by giving the design procedure for the functional filter. Using the unbiasedness conditions, the problem is transformed into a singular problem. A LMI approach is then applied to optimize the gain implemented in the filter. Section 5 summarizes the design algorithm presentation. Numerical example is presented in section 6 and the last section concludes the paper.

2. RELATED WORK

This paper presents a new design of functional H_∞ filter for linear singular system with unknown variable time delay. Comparing with recent references (Ezzine, 2011; Khadhraoui, 2014, 2016) also (Koeing, 2006) our approach offers two contributions; Firstly, we estimate a functional state vector by reducing mathematical conditions and then reduce time calculation. Secondly the proposed H_∞ filter of the unknown inputs systems, with unknown variable time delay can estimate also a part of the unknown inputs vectors. These contributions are of major importance for control purposes and faults detection.

3. PROBLEM FORMULATION

Consider the following continuous-time linear singular delayed system described by:

$$E\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) + Bu(t) + B_d u(t - \tau(t)) + F_1 v(t) + G_1 w(t) \quad (1a)$$

$$y(t) = Cx(t) + Du(t) + F_2 v(t) + G_2 w(t) \quad (1b)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the known input and the output vectors respectively. $v \in \mathbb{R}^q$ and $w \in \mathbb{R}^k$ summarises respectively all the components of the unknown input and of the bounded disturbances that act on dynamic and output. $E, A, A_d, B, B_d, F_1, F_2, G_1, G_2, D$ and C are known matrices of appropriate dimensions.

For simplicity and without loss of generality, we assume that the variable delays on state and input are identical. This delay satisfies:

$$0 \leq \tau(t) \leq \tau_m \quad (2)$$

with τ_m the upper bound of $\tau(t)$.

In the sequel we suppose that (Zasadzinski, 1998):

$$\text{rank}(E) = \bar{n} < n$$

$$\text{rank} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = q$$

$$\text{rank} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = k$$

Then, there exist an orthogonal matrix $R \in \mathbb{R}^{p \times p}$ and a non singular matrix $R \in \mathbb{R}^{q \times q}$, such that:

$$R^T F_2 S = \begin{pmatrix} I_{\bar{q}} & 0_{12} \\ 0_{21} & 0_{22} \end{pmatrix} \quad (3)$$

with $0_{12} \in \mathbb{R}^{\bar{q} \times (q-\bar{q})}$, $0_{21} \in \mathbb{R}^{(p-\bar{q}) \times \bar{q}}$ and $0_{22} \in \mathbb{R}^{(p-\bar{q}) \times (q-\bar{q})}$ such that 0_{xy} is a matrix the component of which are all equal to zero.

By multiplying (1.b) by R^T we obtain:

$$y_1(t) = C_1 x(t) + D_1 u(t) + v_1(t) + G_{21} w(t) \quad (4a)$$

$$y_2(t) = C_2 x(t) + D_2 u(t) + G_{22} w(t) \quad (4b)$$

where:

$$R^T y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ with } y_1 \in \mathbb{R}^{\bar{q}} \text{ and } y_2 \in \mathbb{R}^{p-\bar{q}} \quad (5a)$$

$$R^T C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \text{ with } C_1 \in \mathbb{R}^{\bar{q} \times n} \text{ and } C_2 \in \mathbb{R}^{(p-\bar{q}) \times n} \quad (5b)$$

$$R^T D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \text{ with } D_1 \in \mathbb{R}^{\bar{q} \times m} \text{ and } D_2 \in \mathbb{R}^{(p-\bar{q}) \times m} \quad (5c)$$

and:

$$R^T G_2 = \begin{bmatrix} G_{21} \\ G_{22} \end{bmatrix} \text{ with } G_{21} \in \mathbb{R}^{\bar{q} \times k} \text{ and } G_{22} \in \mathbb{R}^{(p-\bar{q}) \times k} \quad (5d)$$

for:

$$R^T F_2 S S^{-1} = \begin{pmatrix} I_{\bar{q}} & 0_{12} \\ 0_{21} & 0_{22} \end{pmatrix} S^{-1} \quad (6)$$

Similarly, Equation (1a) becomes:

$$E\dot{x}(t) = \bar{A}x(t) + A_d x(t - \tau(t)) + \bar{B}u(t) + B_d u(t - \tau(t)) + F_{11}y_1(t) + F_{12}v_2(t) + \bar{G}w(t) \quad (7)$$

where:

$$F_1 S = \begin{bmatrix} F_{11} & F_{12} \end{bmatrix}, \bar{A} = A - F_{11}C_1, \bar{B} = B - F_{11}D_1, \bar{G} = G_1 - F_{11}G_{21} \quad (8)$$

with $\bar{A} \in \mathbb{R}^{n \times n}$, $\bar{B} \in \mathbb{R}^{n \times m}$, $\bar{G} \in \mathbb{R}^{n \times k}$, $F_{11} \in \mathbb{R}^{n \times \bar{q}}$ and $F_{12} \in \mathbb{R}^{n \times (q-\bar{q})}$ and writing:

$$S^{-1}v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \text{ with } v_1(t) \in \mathbb{R}^{\bar{q}} \text{ and } v_2(t) \in \mathbb{R}^{q-\bar{q}} \quad (9)$$

So:

$$R^T F_2 v(t) = \begin{pmatrix} I_{\bar{q}} & 0_{12} \\ 0_{21} & 0_{22} \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ 0_{q-\bar{q}} \end{pmatrix} \quad (10)$$

Then system (1) becomes:

$$E\dot{x}(t) = \bar{A}x(t) + A_d x(t - \tau(t)) + \bar{B}u(t) + B_d u(t - \tau(t)) + F_{11}y_1(t) + F_{12}v_2(t) + \bar{G}w(t) \quad (11a)$$

$$y_1(t) = C_1 x(t) + D_1 u(t) + v_1(t) + G_{21}w(t) \quad (11b)$$

$$y_2(t) = C_2 x(t) + D_2 u(t) + G_{22}w(t) \quad (11c)$$

3.1. Objective

We aim to design a functional filter for linear singular delayed system. The considered system is affected by a bounded variable time delay acting on the state and the known input vectors. The functional filter estimates a part of the unknown input $v_1(t)$ and a linear combination of the state vector components x_i , $i = 1 \dots n$, using only the input and output measurements. This filter is defined by:

$$z(t) = \begin{pmatrix} Lx(t) \\ v_1(t) \end{pmatrix} \in \mathbb{R}^{r+\bar{q}} \quad (12)$$

with $L \in \mathbb{R}^{r \times n}$.

By replacing $v_1(t)$ by its expression derived from (11b), $z(t)$ can be written as:

$$z(t) = \bar{L}x(t) + \bar{D}_1 u(t) + \bar{G}_{21} w(t) + \bar{I} y_1(t) \quad (13)$$

with:

$$\bar{L} = \begin{bmatrix} L \\ -C_1 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 0_{r \times m} \\ -D_1 \end{bmatrix}, \bar{G}_{21} = \begin{bmatrix} 0_{r \times k} \\ -G_{21} \end{bmatrix}, \bar{I} = \begin{bmatrix} 0_{r \times \bar{q}} \\ \bar{I}_{\bar{q}} \end{bmatrix} \quad (14)$$

with $\bar{L} \in \mathbb{R}^{(r+\bar{q}) \times n}$, $rank(\bar{L}) = r + \bar{q} < n$.

The functional H_∞ filter estimate $z(t)$ described by Equation (13) use only the system known input $u(t)$ and output $y(t)$ as shown in Figure 1.

Figure 1. The functional H_∞ filter design

In the sequel of the paper we suppose that (Darouach, 1995):

$$\text{rank} \begin{pmatrix} E \\ C_2 \\ \bar{L} \end{pmatrix} = \text{rank} \begin{pmatrix} E \\ C_2 \end{pmatrix} \quad (15)$$

Then there exists a non singular matrix T :

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (16)$$

such as:

$$aE + bC_2 = \bar{L} \quad (17)$$

$$cE + dC_2 = 0_{(p-\bar{q}) \times n} \quad (18)$$

with $a \in \mathbb{R}^{(r+\bar{q}) \times n}$, $b \in \mathbb{R}^{(r+\bar{q}) \times (p-\bar{q})}$, $c \in \mathbb{R}^{(p-\bar{q}) \times n}$ and $d \in \mathbb{R}^{(p-\bar{q}) \times (p-\bar{q})}$.

4. TIME DOMAIN DESIGN

The purpose of the paper is to design a functional filter of the system defined by (11) and (13) of the form:

$$\dot{\eta}(t) = M\eta(t) + M_d \eta(t - \tau(t)) + Ku(t) + K_d u(t - \tau(t)) \quad (19a)$$

$$\hat{z}(t) = \eta(t) + Pu(t) + H_1 y_1(t) + H_2 y_2(t) \quad (19b)$$

where $\eta(t) \in \mathbb{R}^{r+\bar{q}}$ is the state vector of the filter and $\hat{z}(t)$ is the estimate of the functional state $\bar{z}(t) \in \mathbb{R}^{r+\bar{q}}$. The matrices $M, M_d, K, K_d, N_1, N_2, N_d, P, H_1$ and H_2 will be determined using the LMI approach.

4.1. Conditions of the Functional Filter Synthesis

Let's define $e(t)$ as the time estimation error:

$$e(t) = z(t) - \hat{z}(t) \quad (20)$$

Using (13) and (19b), $e(t)$ is given by:

$$e(t) = (\bar{L} - H_2 C_2)x(t) + (\bar{D}_1 - P - H_2 D_2)u(t) + (\bar{G}_{21} - H_2 G_{22})w(t) \quad (21)$$

We assume that:

$$P = \bar{D}_1 - H_2 D_2 \quad (22)$$

$$H_1 = \bar{I} \quad (23)$$

So the Equation (21) is equivalent to:

$$e(t) = (\bar{L} - H_2 C_2)x(t) + (\bar{G}_{21} - H_2 G_{22})w(t) - \eta(t) \quad (24)$$

Using (17) and (18) we can write:

$$e(t) = (aE + bC_2 - H_2 C_2)x(t) + \beta(cE + dC_2)x(t) + (\bar{G}_{21} - H_2 G_{22})w(t) - \eta(t) \quad (25)$$

$$= (a + \beta c)Ex(t) + (b + \beta d - H_2)C_2x(t) + (\bar{G}_{21} - H_2 G_{22})w(t) - \eta(t)$$

where $\beta \in \mathbb{R}^{(r+\bar{q}) \times (p-\bar{q})}$.

We assume that:

$$H_2 = b + \beta d \quad (26)$$

So Equation (25) can be written in the following form:

$$e(t) = \varphi Ex(t) - \eta(t) + (\bar{G}_{21} - H_2 G_{22})w(t) \quad (27)$$

with:

$$\varphi = a + \beta c, \varphi \in \mathbb{R}^{(r+\bar{q}) \times n} \quad (28)$$

The problem of the functional filter is to determine $M, M_d, K, K_d, N_1, N_2, N_d, P, H_1$ and H_2 such that:

$$\lim_{t \rightarrow \infty} e(t) = 0, \text{ if } w = 0$$

The filtering error (27) is asymptotically stable and satisfy the H_∞ performance.

The H_∞ criterion is given by:

$$0 < H_{\epsilon w \infty} = \sup_{w \neq 0} \frac{\epsilon_2}{w_2} < \gamma \quad (29)$$

with $H_{\epsilon w}(s) = \frac{\epsilon(s)}{w(s)}$ is a transfer matrix and γ is a positive scalar.

For this, we propose the following theorem.

4.1.1. Theorem 1

The unbiasedness estimation error given by (27) relative to system (11), (13) and functional H_∞ filter (19) is verified such that:

$$\dot{e}(t) = Me(t) + M_d e(t - \tau(t)) + \alpha w(t) + \varsigma w(t - \tau(t)) + \delta \dot{w}(t) \quad (30)$$

with:

$$\alpha = \varphi \bar{G} - M(\bar{G}_{21} - H_2 G_{22}) N_2 G_{22} \quad (31)$$

$$\varsigma = -M_d(\bar{G}_{21} - H_2 G_{22}) N_d G_{22} \quad (32)$$

$$\delta = \bar{G}_{21} - H_2 G_{22} \quad (33)$$

α , ς and $\delta \in \mathbb{R}^{(r+\bar{q}) \times k}$ if and only if the following Equations are satisfied:

$$\varphi \bar{A} - N_2 C_2 - M \varphi E = 0_{(r+\bar{q}) \times n}$$

$$\varphi A_d - N_d C_2 - M_d \varphi E = 0_{(r+\bar{q}) \times n}$$

$$\varphi \bar{B} - N_2 D_2 = K$$

$$\varphi B_d - N_d D_2 = K_d$$

$$\varphi F_{11} = N_1$$

$$\varphi F_{12} = 0_{(r+\bar{q}) \times (p-\bar{q})}$$

4.1.2. Proof 1

The time derivative of $e(t)$ given in (27) is:

$$\dot{e}(t) = \varphi E \dot{x}(t) - \dot{\eta}(t) + (\bar{G}_{21} - H_2 G_{22}) \dot{w}(t) \quad (34)$$

Replacing $E \dot{x}(t)$ and $\dot{\eta}(t)$ by their expressions given by (11a) and (19a) respectively, we have:

$$\begin{aligned} \dot{e}(t) = & M e(t) + M_d e(t - \tau(t)) + [\varphi \bar{G} - M(\bar{G}_{21} - H_2 G_{22}) - N_2 G_{22}] w(t) \\ & + [\bar{G}_{21} - H_2 G_{22}] \dot{w}(t) + [-M_d(\bar{G}_{21} - H_2 G_{22}) - N_d G_{22}] w(t - \tau(t)) \\ & + [\varphi \bar{A} - N_2 C_2 - M \varphi E] x(t) + [\varphi A_d - N_d C_2 - M_d \varphi E] x(t - \tau(t)) \\ & + [\varphi \bar{B} - N_2 D_2 - K] u(t) + [\varphi B_d - N_d D_2 - K_d] u(t - \tau(t)) \\ & + [\varphi F_{11} - N_1] y_1(t) + \varphi F_{12} v_2(t) \end{aligned} \quad (35)$$

Using conditions (i)-(vi) of theorem 1, and assuming that $w(t) = 0$, $\dot{e}(t) = M e(t) + M_d e(t - \tau(t))$.
Therefore $\lim_{t \rightarrow \infty} e(t) = 0$.

4.2. Determination of Observer Matrices

At this stage, we propose to design the functional filter dependently from the state delay using a Lyapunov-Krasovskii stability theory (Krasovskii, 1963) and a LMI approach. So, the estimated functional state $\hat{z}(t)$ converges asymptotically to the real one $z(t)$ with the satisfaction of condition (29).

Now, replacing φ given by its expression (28) in conditions i), ii) and vi) of theorem 1 and using (18), we have: By replacing (23) in conditions i) and iv) of theorem 1 and using (13), we have:

$$a\bar{A} = MaE + JC_2 - \beta c\bar{A} \quad (36)$$

$$aA_d = M_d aE + J_d C_2 - \beta c A_d \quad (37)$$

and:

$$aF_{12} = -\beta c F_{12} \quad (38)$$

with:

$$J = N_2 - M\beta d \quad (39)$$

$$J_d = N_d - M_d \beta d \quad (40)$$

Equations (36) - (38) can be written in the following matrix form:

$$\Gamma = \Omega \Theta \quad (41)$$

with:

$$\Omega = \begin{bmatrix} M & M_d & J & J_d & \beta \end{bmatrix} \quad (42)$$

$$\Theta = \begin{pmatrix} \begin{matrix} aE & \bar{0}_1 & \bar{0}_2 \\ \bar{0}_1 & aE & \bar{0}_2 \\ C_2 & \bar{0}_3 & \bar{0}_4 \\ \bar{0}_3 & C_2 & \bar{0}_4 \\ -c\bar{A} & -cA_d & -cF_{12} \end{matrix} \end{pmatrix} \quad (43)$$

$$\Gamma = \begin{bmatrix} a\bar{A} & aA_d & aF_{12} \end{bmatrix} \quad (44)$$

with $\bar{0}_1 \in \mathbb{R}^{(r+\bar{q}) \times n}$, $\bar{0}_2 \in \mathbb{R}^{(r+\bar{q}) \times (q-\bar{q})}$, $\bar{0}_3 \in \mathbb{R}^{(p-\bar{q}) \times n}$ and $\bar{0}_4 \in \mathbb{R}^{(p-\bar{q}) \times (q-\bar{q})}$.

We note that the general Solution (41) exists if and only if (Zasadzinski, 1998):

$$rank \begin{bmatrix} \Theta \\ \Gamma \end{bmatrix} = rank(\Theta) \quad (45)$$

Therefore, by respecting the condition (36):

$$\Omega = \Gamma \Theta^+ - Z \left(\bar{I} - \Theta \Theta^+ \right) \quad (46)$$

where $\bar{I}$ is an identity matrix of dimension $(2r + 3p - \bar{q})$, Θ^+ is the generalized inverse of matrix Θ and $Z \in \mathbb{R}^{(r+\bar{q}) \times (2r+3p-\bar{q})}$ is an arbitrary matrix, that will be determined later using the LMI approach.

The unknown matrix M is defined by:

$$M = \Omega \begin{pmatrix} I_{r+\bar{q}} \\ \bar{0}_1 \\ \bar{0}_2 \\ \bar{0}_2 \\ \bar{0}_2 \end{pmatrix} = M_1 + ZM_2 \quad (47)$$

$$0_1 \in \mathbb{R}^{(r+\bar{q}) \times (r+\bar{q})} \text{ and } 0_2 \in \mathbb{R}^{(r+\bar{q}) \times (p-\bar{q})}$$

By replacing Ω by its expression given by (46) in (47), we obtain:

$$M_1 = \Gamma \Theta^+ \begin{pmatrix} I_{r+\bar{q}} \\ 0_1 \\ 0_2 \\ 0_2 \\ 0_2 \end{pmatrix}, M_2 \begin{pmatrix} I_{r+\bar{q}} \\ 0_1 \\ 0_2 \\ 0_2 \\ 0_2 \end{pmatrix} \quad (48)$$

The unknown matrix M_d is defined by:

$$M = \Omega \begin{pmatrix} 0_1 \\ I_{r+\bar{q}} \\ 0_2 \\ 0_2 \\ 0_2 \end{pmatrix} = M_{d1} + ZM_{d2} \quad (49)$$

with:

$$M_{d1} = \Gamma \Theta^+ \begin{pmatrix} 0_1 \\ I_{r+\bar{q}} \\ 0_2 \\ 0_2 \\ 0_2 \end{pmatrix}, M_{d2} \begin{pmatrix} I_{r+\bar{q}} \\ 0_2 \\ 0_2 \\ 0_2 \end{pmatrix} \quad (50)$$

Similarly, J is obtained such:

$$J = \Omega \begin{pmatrix} 0_1 \\ 0_1 \\ \bar{I}_1 \\ 0_2 \\ 0_2 \end{pmatrix}, = J_1 + ZJ_2 \quad (51)$$

$$\bar{I}_1 \in \mathbb{R}^{(p-\bar{q}) \times (r+\bar{q})}$$

with:

$$J_1 = \Gamma \Theta^+ \begin{pmatrix} 0_1 \\ 0_1 \\ \bar{I}_1 \\ 0_2 \\ 0_2 \end{pmatrix}, M_2 \begin{pmatrix} = \\ I - \Theta \Theta^+ \end{pmatrix} \begin{pmatrix} 0_1 \\ 0_1 \\ \bar{I}_1 \\ 0_2 \\ 0_2 \end{pmatrix} \quad (52)$$

J_d is obtained such:

$$J_d = \Omega \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ \bar{I}_1 \\ 0_2 \end{pmatrix}, = J_{d1} + Z J_{d2} \quad (53)$$

with:

$$J_{d1} = \Gamma \Theta^+ \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ \bar{I}_1 \\ 0_2 \end{pmatrix}, M_2 \begin{pmatrix} = \\ I - \Theta \Theta^+ \end{pmatrix} \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ \bar{I}_1 \\ 0_2 \end{pmatrix} \quad (54)$$

and β is obtained such:

$$\beta = \Omega \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ 0_2 \\ \bar{I}_1 \end{pmatrix} = \beta_1 + Z \beta_2 \quad (55)$$

with:

$$\beta_1 = \Gamma \Theta^+ \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ 0_2 \\ \bar{I}_1 \end{pmatrix}, \beta_2 \begin{pmatrix} = \\ I - \Theta \Theta^+ \end{pmatrix} \begin{pmatrix} 0_1 \\ 0_1 \\ 0_2 \\ 0_2 \\ \bar{I}_1 \end{pmatrix} \quad (56)$$

Using (39), (47) and (55) we can write $N_2 G_{22}$ such that:

$$\begin{aligned} N_2 G_{22} &= (J + M\beta d) G_{22} \\ &= J G_{22} + (M_1 + ZM_2)(\beta_1 + Z\beta_2) d G_{22} \\ &= J G_{22} + M_1(\beta_1 + Z\beta_2) d G_{22} + (ZM_2\beta_1 + ZM_2Z\beta_2) d G_{22} \end{aligned} \quad (57)$$

Similarly using (40), (49) and (55) $N_d G_{22}$ can be written as:

$$\begin{aligned} N_d G_{22} &= (J_d + M_d\beta d) G_{22} \\ &= J_d G_{22} + (M_{d1} + ZM_{d2})(\beta_1 + Z\beta_2) d G_{22} \\ &= J_d G_{22} + M_{d1}(\beta_1 + Z\beta_2) d G_{22} + (ZM_{d2}\beta_1 + ZM_{d2}Z\beta_2) d G_{22} \end{aligned} \quad (58)$$

The matrices $N_2 G_{22}$ and $N_d G_{22}$ have a bi-linearity (ZM_2Z) and $(ZM_{d2}Z)$. In order to avoid this bi-linearity we assume that the gain matrix Z satisfies the following relation:

$$Z\beta_2 = 0_{(r+\bar{q}) \times (p-\bar{q})} \quad (59)$$

So there always exists a matrix $Z_1 \in \mathbb{R}^{(r+\bar{q}) \times (2r+3p-\bar{q})}$ such that:

$$Z = Z_1 \begin{pmatrix} I \\ -\beta_2\beta_2^+ \end{pmatrix} = Z_1 \Delta \quad (60)$$

At this stage we avoid the gain Z and we consider the new gain Z_1 . Then M , M_d , J , J_d and β are defined as:

$$M = M_1 + Z_1 \Delta M_2 \quad (61)$$

$$M_d = M_{d1} + Z_1 \Delta M_{d2} \quad (62)$$

$$J = J_1 + Z_1 \Delta J_2 \quad (63)$$

$$J_d = J_{d1} + Z_1 \Delta J_{d2} \quad (64)$$

$$\beta = \beta_1 \quad (65)$$

Rewriting (31) - (33) and using (26) and (61) - (64) we can deduct that:

$$\alpha = \alpha_1 + Z_1 \Delta \alpha_2 \quad (66)$$

$$\varsigma = \varsigma_1 + Z_1 \Delta \varsigma_2 \quad (67)$$

$$\delta = \bar{G}_{21} - b G_{22} - \beta_1 d G_{22} \quad (68)$$

with:

$$\alpha_1 = \varphi \bar{G} - M_1 (\bar{G}_{21} - b G_{22}) - J_1 G_{22} \quad (69)$$

$$\alpha_2 = -M_2 (\bar{G}_{21} - b G_{22}) - J_2 G_{22} \quad (70)$$

$$\varsigma_1 = -M_{d1} (\bar{G}_{21} - b G_{22}) - J_{d1} G_{22} \quad (71)$$

$$\varsigma_2 = -M_{d2} (\bar{G}_{21} - b G_{22}) - J_{d2} G_{22} \quad (72)$$

Hence all filter matrices are determined if and only if the matrix Z_1 is known which will be proposed in the following.

The error dynamics given by (30) contains the time derivative of $\dot{w}(t)$. This problem is generally solved by adding an additional constraint on the filter matrices or by choosing a new type of norm for the system. Based on (Khadhraoui, 2014) we propose to rewrite (30) into a singular form. Then the singular Equation is given:

$$\begin{bmatrix} I_{r+\bar{q}} & -\delta \\ \bar{0}_{21} & \bar{0}_{22} \end{bmatrix} \begin{bmatrix} \dot{e}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} M & \alpha \\ \bar{0}_{21} & -I_k \end{bmatrix} \begin{bmatrix} e(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} M_d & \delta \\ \bar{0}_{21} & \bar{0}_{22} \end{bmatrix} \begin{bmatrix} e(t - \tau(t)) \\ w(t - \tau(t)) \end{bmatrix} + \begin{bmatrix} \bar{0}_{21}^T \\ I_k \end{bmatrix} w(t) \quad (73)$$

with $\bar{0}_{21} \in \mathbb{R}^{k \times (r+\bar{q})}$ and $\bar{0}_{22} \in \mathbb{R}^{k \times k}$.

Let's consider that:

$$l = r + \bar{q} + k, \epsilon(t) = \begin{bmatrix} e(t) \\ w(t) \end{bmatrix} \in \mathbb{R}^l \quad (74)$$

Then we have:

$$\rho \dot{\epsilon}(t) = \tilde{M} \epsilon(t) + \tilde{M}_d \epsilon(t - \tau(t)) + \tilde{K} w(t) \quad (75)$$

where:

$$\rho = \begin{bmatrix} I_{r+\bar{q}} & -\delta \\ \bar{0}_{21} & \bar{0}_{22} \end{bmatrix} \quad (76)$$

$$\tilde{M} = \begin{bmatrix} M & \alpha \\ \bar{0}_{21} & -I_k \end{bmatrix} \quad (77)$$

and:

$$\tilde{M}_d = \begin{bmatrix} M_d & \delta \\ \bar{0}_{21} & \bar{0}_{22} \end{bmatrix} \quad (78)$$

$$\tilde{K} = \begin{bmatrix} \bar{0}_{21}^T \\ I_k \end{bmatrix} \quad (79)$$

with $\rho, \tilde{M}, \tilde{M}_d \in \mathbb{R}^{l \times l}$ and $\tilde{K} \in \mathbb{R}^{l \times k}$.

A LMI approach is proposed in the following theorem to determine the gain Z_1 which parametrizes the H_∞ filter matrices.

4.2.1. Theorem 2

System (19) is a functional filter for system (1) if there exist matrices $P_1 = P_1^T > 0, P_2, P_3 = P_3^T > 0$ and Y solution of the following LMI that holds for a given $\gamma > 0$:

$$\begin{pmatrix} \psi & X \\ * & \Pi \end{pmatrix} < 0 \quad (80)$$

and:

$$\tilde{P} \rho = \rho^T \tilde{P}^T > 0 \quad (81)$$

$$\tilde{P} = \begin{pmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{pmatrix} \quad (82)$$

$$P_2 = -\delta^T P_1 \quad (83)$$

where δ and ρ are given by (68) and (76) and the matrices ψ , X and Π are given in the Appendix.

The gain Z_1 is given by:

$$Z_1 = P_1^{-1}Y \quad (84)$$

4.2.2. Proof 2

To prove the Theorem 2 we consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \quad (85)$$

with:

$$V_1(t) = \epsilon(t)^T \tilde{P} \rho \epsilon(t) \quad (86)$$

$$V_2(t) = \int_0^{\tau(t)} \int_{t-\theta}^t \epsilon(\mu)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(\mu) d\mu d\theta \quad (87)$$

$$V_3(t) = \int_0^{\tau(t)} \int_{t-\theta}^t \epsilon(\mu)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(\mu) d\mu d\theta \quad (88)$$

where $\tilde{P} \in \mathbb{R}^{l \times l}$ is symmetric positive and definite matrix and $\epsilon(t)$ is defined in (74).

According to condition (2), $V(t) \geq 0$, $\forall t \in \mathbb{R}^+$ and knowing that $\epsilon(\mu)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(\mu)$ and $\epsilon(\mu)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(\mu)$ are positive scalars, we can write:

$$0 \leq V(t) \leq \bar{V}(t) \quad (89)$$

with:

$$\bar{V}(t) = V_1(t) + \tau_m V_{21}(t) + \tau_m V_{31}(t) \quad (90)$$

$$V_{21}(t) = \int_0^{\tau_m} \int_{t-\theta}^t \epsilon(\mu)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(\mu) d\mu d\theta \quad (91)$$

$$V_{31}(t) = \int_0^{\tau_m} \int_{t-\theta}^t \epsilon(\mu)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(\mu) d\mu d\theta \quad (92)$$

Based on (89), if $\lim_{t \rightarrow \infty} \bar{V}(t) = 0$ then $\lim_{t \rightarrow \infty} V(t) = 0$.

To prove that $\bar{V}(t) \approx 0$ when $t \rightarrow \infty$ we can prove that $\bar{V}(t) \approx 0$ when $t \rightarrow \infty$ (see (Khadhraoui, 2014)).

The time derivative of the functional $\bar{V}(t)$ is:

$$\dot{\bar{V}}(t) = \dot{V}_1(t) + \tau_m \dot{V}_{21}(t) + \tau_m \dot{V}_{31}(t) \quad (93)$$

where:

$$\dot{V}_1(t) = \dot{\epsilon}(t)^T \rho^T \tilde{P} \epsilon(t) + \epsilon(t)^T \tilde{P} \rho \dot{\epsilon}(t) \quad (94)$$

Using Equation (75), we deduct that:

$$\dot{V}_1(t) = \epsilon(t)^T [\tilde{M}^T \tilde{P} + \tilde{P} \tilde{M}] \epsilon(t) + \epsilon(t)^T \tilde{P} \tilde{K} w(t) + \epsilon(t)^T \tilde{P} \tilde{M}_d \epsilon(t - \tau(t)) \quad (95)$$

$$+ w(t)^T \tilde{K}^T \tilde{P} \epsilon(t) + \epsilon(t - \tau(t))^T \tilde{M}_d^T \tilde{P} \epsilon(t)$$

Moreover:

$$\dot{V}_{21}(t) = \tau_m \epsilon(t)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(t) - \int_0^{\tau_m} \epsilon(t - \theta)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(t - \theta) d\theta \quad (96)$$

and:

$$\dot{V}_{31}(t) = \tau_m \epsilon(t)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(t) - \int_0^{\tau_m} \epsilon(t - \theta)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(t - \theta) d\theta \quad (97)$$

Let's define that:

$$\xi(t - \theta) = \tilde{M} \epsilon(t - \theta) \quad (98)$$

and:

$$\xi_d(t - \theta) = \tilde{M}_d \epsilon(t - \theta) \quad (99)$$

with ξ and $\xi_d \in \mathbb{R}^l$.

So we can write:

$$\dot{V}_{21}(t) = \tau_m \epsilon(t)^T \tilde{M}^T \tilde{P} \tilde{M} \epsilon(t) - \int_0^{\tau_m} \xi(t - \theta)^T \tilde{P} \xi(t - \theta) d\theta \quad (100)$$

and:

$$\dot{V}_{31}(t) = \tau_m \epsilon(t)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \epsilon(t) - \int_0^{\tau_m} \xi_d(t - \theta)^T \tilde{P} \xi_d(t - \theta) d\theta \quad (101)$$

As θ is finite, the quantities $\xi(t - \theta)$ and $\xi_d(t - \theta)$, given by (98) and (99) respectively, satisfy:

$$\lim_{t \rightarrow \infty} \xi(t - \theta) = \lim_{t \rightarrow \infty} \xi(t) \quad (102)$$

and:

$$\lim_{t \rightarrow \infty} \xi_d(t - \theta) = \lim_{t \rightarrow \infty} \xi_d(t) \quad (103)$$

Therefore:

$$\lim_{t \rightarrow \infty} \int_0^{\tau_m} \xi(t - \theta)^T \tilde{P} \xi(t - \theta) d\theta = \tau_m \lim_{t \rightarrow \infty} \xi(t)^T \tilde{P} \xi(t) \quad (104)$$

and:

$$\lim_{t \rightarrow \infty} \int_0^{\tau_m} \xi_d(t - \theta)^T \tilde{P} \xi_d(t - \theta) d\theta = \tau_m \lim_{t \rightarrow \infty} \xi_d(t)^T \tilde{P} \xi_d(t) \quad (105)$$

Then we suppose that:

$$\lambda = \lim_{t \rightarrow \infty} \xi(t) \quad (106)$$

$$v = \lim_{t \rightarrow \infty} \xi_d(t) \quad (107)$$

$$\varepsilon = \lim_{t \rightarrow \infty} \epsilon(t) \quad (108)$$

Uniform asymptotic stability implies that:

$$\lim_{t \rightarrow \infty} \dot{\bar{V}}(t) \leq 0 \quad (109)$$

According to (93)-(108) the Equation (93) is equivalent to:

$$\lim_{t \rightarrow \infty} \dot{\bar{V}}(t) = \varepsilon(t)^T \left[(\tilde{M} + \tilde{M}_d)^T \tilde{P} + \tilde{P}(\tilde{M} + \tilde{M}_d) \right] \varepsilon(t) + \tau_m^2 \varepsilon(t)^T \tilde{M}^T \tilde{P} \tilde{M} \varepsilon(t) \quad (110)$$

$$+ \tau_m^2 \varepsilon(t)^T \tilde{M}_d^T \tilde{P} \tilde{M}_d \varepsilon(t) + w(t)^T \tilde{K}^T \tilde{P} \varepsilon(t) + \varepsilon(t)^T \tilde{P} \tilde{K} w(t) - \tau_m^2 \lambda^T \tilde{P} \lambda - \tau_m^2 v^T \tilde{P} v$$

In order to establish sufficient conditions for existence of (19) according to H_∞ criterion, we should verify the following inequality (Khadhraoui, 2016):

$$H(\varepsilon, t) = \dot{V}(t) + \varepsilon(t)^T \varepsilon(t) - \gamma^2 w(t)^T w(t) < 0 \quad (111)$$

which can be written as follows:

$$\begin{bmatrix} \varepsilon^T & \lambda^T & v^T & w^T \end{bmatrix} \Psi \begin{bmatrix} \varepsilon \\ \lambda \\ v \\ w \end{bmatrix} \quad (112)$$

with:

$$\Psi = \begin{pmatrix} Q & 0_{l \times l} & 0_{l \times l} & \tilde{P} \tilde{K} \\ * & -\tau_m^2 \tilde{P} & 0_{l \times l} & 0_{l \times k} \\ * & * & -\tau_m^2 \tilde{P} & 0_{l \times k} \\ * & * & * & -\gamma^2 I_k \end{pmatrix} \quad (113)$$

and:

$$Q = (\tilde{M} + \tilde{M}_d)^T \tilde{P} + \tilde{P}(\tilde{M} + \tilde{M}_d) + \tau_m^2 \tilde{M}^T \tilde{P} \tilde{M} + \tau_m^2 \tilde{M}_d^T \tilde{P} \tilde{M}_d + I_l \quad (114)$$

To avoid the quadratic form present in Equation (113), we propose to rewrite this Equation as following form:

$$\Psi = U - \Phi^T \Xi^{-1} \Phi \quad (115)$$

with:

$$\Phi = \begin{pmatrix} \tau_m \tilde{M} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ 0_{l \times l} & I_l & 0_{l \times l} & 0_{l \times k} \\ \tau_m \tilde{M}_d & 0_{l \times l} & I_l & 0_{l \times k} \\ 0_{k \times l} & 0_{k \times l} & 0_{k \times l} & \gamma I_k \end{pmatrix} \quad (116)$$

$$\Xi = - \begin{pmatrix} \tilde{P}^{-1} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & \tilde{P}^{-1} & 0_{l \times l} & 0_{l \times k} \\ * & * & \tilde{P}^{-1} & 0_{l \times k} \\ * & * & * & I_k \end{pmatrix} \quad (117)$$

and:

$$U = \begin{pmatrix} Q_l & 0_{l \times l} & -\tau_m \tilde{M}_d^T \tilde{P} & \tilde{P} \tilde{K} \\ * & -(\tau_m^2 + 1) \tilde{P} & 0_{l \times l} & 0_{l \times k} \\ * & * & -(\tau_m^2 + 1) \tilde{P} & 0_{l \times k} \\ * & * & * & -2\gamma^2 I_k \end{pmatrix} \quad (118)$$

$$Q_l = (\tilde{M} + \tilde{M}_d)^T \tilde{P} + \tilde{P}(\tilde{M} + \tilde{M}_d) + I_l \quad (119)$$

According to the Schur lemma, $\Psi < 0$ and $\Xi < 0$ if and only if:

$$\Lambda = \begin{bmatrix} \Xi & \Phi \\ \Phi^T & U \end{bmatrix} < 0 \quad (120)$$

Now we propose to apply a congruence transformation (Khadhraoui, 2014) to Λ such as:

$$\Sigma^T \Lambda \Sigma < 0 \quad (121)$$

where Σ is a non singular matrix given by:

$$\Sigma = \begin{pmatrix} \tilde{P} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} & 0_{l \times l} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & \tilde{P} & 0_{l \times l} & 0_{l \times k} & 0_{l \times l} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & * & \tilde{P} & 0_{l \times k} & 0_{l \times l} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & * & * & I_k & 0_{k \times l} & 0_{k \times l} & 0_{k \times l} & 0_{k \times k} \\ * & * & * & * & I_l & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & * & * & * & * & I_l & 0_{l \times l} & 0_{l \times k} \\ * & * & * & * & * & * & I_l & 0_{l \times k} \\ * & * & * & * & * & * & * & \tilde{P} \end{pmatrix} \quad (122)$$

Then by replacing $\tilde{M}$, $\tilde{M}_d$ and $\tilde{K}$ by their expressions given by (77), (78) and (79) respectively in (56), theorem 2 holds.

5. ALGORITHM FOR FUNCTIONAL FILTER DESIGN

Step 1: Check that conditions 1), 2), 3) and 4) are satisfied.

Step 2: Compute matrices R and S using (3).

Step 3: Check that condition (15) is verified.

Step 4: Compute matrix T using (17) and (18).

Step 5: Compute Θ and Γ using (43) and (44).

Step 6: Deduce the values of matrices M_1 , M_2 , M_{d1} , M_{d2} , J_1 , J_2 , J_{d1} , J_{d2} , β_1 and β_2 from (47)-(56).

Step 7: Verify that the condition (45) is satisfied. Then resolution of the proposed LMI (80), gives the gain matrix Z_1 .

Step 8: Compute M , M_d , J , J_d and β using (61)-(65).

Step 9: Get matrices P , H_1 , H_2 , N_2 and N_d using (22), (23), (26), (39) and (40).

Step 10: Get matrices K , K_d and N_1 using iii) - v) from theorem 1.

6. NUMERICAL EXAMPLES

Let's consider system (1) where:

$$E = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, A_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, G_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, G_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, L = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

The variable state and known input delay is a sinusoid such:

$$\tau(t) = -0.5 \sin(1.2t) + 1$$

According to (2), we have $0.5 \leq \tau(t) \leq 1.5, \forall t \in \mathbb{R}^+$.

The H_∞ criterion is bounded by $\gamma = 0.7$.

We suppose that $u(t)$, $w(t)$ and $u(t)$ are respectively given by Figures 2-4.

Figure 2. Input signal $u(t)$

Figure 3. Bounded disturbance $w(t)$

Figure 4. Unknown inputs $v(t)$

Following the procedure proposed below, we obtain:

$$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, S = \begin{pmatrix} -0.2 & -0.894 \\ -0.4 & 0.447 \end{pmatrix}$$

By multiplying the unknown inputs $v(t)$ by S^{-1} we have (Figure 5):

Figure 5. Part of unknown inputs $v_1(t)$

The resolution of LMI (80) gives:

$$P_1 = \begin{pmatrix} 51.4 & 51.37 \\ 51.37 & 51.4 \end{pmatrix}, P_2 = \begin{pmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{pmatrix}, P_3 = 0.708$$

$$Y = \begin{pmatrix} -20.79 & -20.79 & -14.97 & -14.97 & 0 & 0 & 0 \\ -20.79 & -20.79 & -14.97 & -14.97 & 0 & 0 & 0 \end{pmatrix}$$

$$Z_1 = \begin{pmatrix} -0.2 & -0.2 & -0.14 & -0.14 & 0 & 0 & 0 \\ -0.2 & -0.2 & -0.14 & -0.14 & 0 & 0 & 0 \end{pmatrix}$$

So, the functional filter matrices values are given as follows:

$$M = \begin{pmatrix} -5.20 & 4.79 \\ 4.79 & -5.20 \end{pmatrix}, M_d = \begin{pmatrix} 0.850 & -1.14 \\ -1.14 & 0.85 \end{pmatrix}, N_1 = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}, N_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, N_d = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$$

$$K = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, K_d = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}, H_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, H_2 = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}, P = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

Next figures are carried out using the Simulink platform of Matlab. Figure 6 shows a comparison between the real and the estimated functional state vector.

Figure 6. Functional state $z(t)$

Figure 7 represents the real and the estimated values of part $v_1(t)$.

We can remark as it shown in Figure 6 and Figure 7 the time domain behavior of the functional H_∞ filter, so it has a rapid convergence dynamic. The effect of the disturbance on the estimation error is evaluated by $H_{ew\infty} = 0,52 < \gamma$

Figure 7. Unknown inputs $v_1(t)$

Then the estimation error of the functional state $z(t)$ and the unknown input $v_1(t)$ are given respectively by Figure 8 and Figure 9.

We note that the error estimation achieves the permanent phase at $t = 6s$ with respect to the H_∞ disturbance attenuation criterion. So, we highlight the effectiveness of the design techniques dependently on the delay with an application on a variable state delay.

Figure 8. Estimation error of functional state

7. CONCLUSION

In this paper, we have proposed a new method for the design functional H_∞ filter of linear singular systems with bounded disturbance, unknown inputs and bounded variable time delay acting on the state and the known input vectors. Using only the input and output measurements, the designed filter will estimate a reduced functional state and a part of the unknown inputs, which can be used on control purposes and handle fault detection.

Figure 9. Estimation error of unknown input

This procedure is based on the unbiasedness of the estimation error, the Lyapunov-Krasovskii stability theory and the disturbance minimizing effect on the estimation error.

Such results are of major importance for fault detection and control using feedback state schema.

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APPENDIX: MATRICES

The matrices ψ , X and Π used in theorem 2 are defined as:

$$\psi = \begin{pmatrix} -\tilde{P} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ * & -\tilde{P} & 0_{l \times l} & 0_{l \times k} \\ * & * & -\tilde{P} & 0_{l \times k} \\ * & * & * & -I_k \end{pmatrix}$$

$$X = \begin{pmatrix} \tau_m \tilde{P} \tilde{M} & 0_{l \times l} & 0_{l \times l} & 0_{l \times k} \\ 0_{l \times l} & \tilde{P} & 0_{l \times l} & 0_{l \times k} \\ \tau_m \tilde{P} \tilde{M}_d & 0_{l \times l} & \tilde{P} & 0_{l \times k} \\ 0_{k \times l} & 0_{k \times l} & 0_{k \times l} & \gamma I_k \end{pmatrix}$$

Replacing $\tilde{M}$, $\tilde{M}_d$ and $\tilde{P}$ by their values in (77), (78) and (82) respectively, the matrix X can be written as:

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} & X_{37} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{47} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} & X_{57} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} & X_{67} \\ X_{71} & X_{72} & X_{73} & X_{74} & X_{75} & X_{76} & X_{77} \end{pmatrix}$$

where:

$$\begin{aligned} X_{11} &= \tau_m P_1 M_1 + \tau_m Y \Delta M_2 \\ X_{12} &= \tau_m P_1 \alpha_1 + \tau_m Y \Delta \alpha_2 + \tau_m P_1 \delta \\ X_{13} &= X_{15} = X_{31} = X_{35} = X_{53} = 0_1 \\ X_{14} &= X_{16} = X_{17} = X_{32} = X_{36} = X_{37} = X_{54} = X_{57} = \bar{0}_{21}^T \\ X_{21} &= -\tau_m \delta^T P_1 M_1 - \tau_m \delta^T Y \Delta M_2 \\ X_{22} &= -\tau_m \delta^T P_1 \alpha_1 - \tau_m \delta^T Y \Delta \alpha_2 - \tau_m P_3 \\ X_{23} &= X_{25} = X_{41} = X_{45} = X_{63} = X_{71} = X_{73} = X_{75} = \bar{0}_{21} \\ X_{24} &= X_{26} = X_{27} = X_{42} = X_{46} = X_{47} = X_{64} = X_{67} = X_{72} = X_{74} = X_{76} = 0_{k \times k} \\ X_{33} &= X_{55} = p_1 \\ X_{34} &= X_{56} = -P_1 \delta \end{aligned}$$

$$\begin{aligned}
 X_{43} &= X_{65} = -\delta^T P_1 \\
 X_{44} &= X_{66} = P_3 \\
 X_{51} &= \tau_m P_1 M_{d1} + \tau_m Y \Delta M_{d2} \\
 X_{52} &= \tau_m P_1 \varsigma_1 + \tau_m Y'' \varsigma_2 \\
 X_{61} &= -\tau_m \delta^T P_1 M_{d1} - \tau_m \delta^T Y'' M_{d2} \\
 X_{62} &= -\tau_m \delta^T P_1 \varsigma_1 - \tau_m \delta^T Y'' \varsigma_2 \\
 X_{77} &= \gamma I_k
 \end{aligned}$$

and:

$$\Pi = \begin{pmatrix} Q_1 & 0_{l \times l} & -\tau_m \tilde{M}_d^T \tilde{P} & \tilde{P} \tilde{K} \\ * & -(\tau_m^2 + 1) \tilde{P} & 0_{l \times l} & 0_{l \times k} \\ * & * & -(\tau_m^2 + 1) \tilde{P} & 0_{l \times k} \\ * & * & * & -2\gamma^2 I_k \end{pmatrix}$$

Similarly by replacing $\tilde{M}_d$, $\tilde{K}$, $\tilde{P}$ and Q_1 by their values in (78), (79), (82) and (119) respectively, the matrix Π can be written as:

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} & \Pi_{17} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} & \Pi_{27} \\ * & * & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} & \Pi_{37} \\ * & * & * & \Pi_{44} & \Pi_{45} & \Pi_{46} & \Pi_{47} \\ * & * & * & * & \Pi_{55} & \Pi_{56} & \Pi_{57} \\ * & * & * & * & * & \Pi_{66} & \Pi_{67} \\ * & * & * & * & * & * & \Pi_{77} \end{pmatrix}$$

$$\Pi_{11} = M_1^T P_1 + M_2^T \Delta^T Y^T + M_{d1}^T P_1 + M_{d2}^T \Delta^T Y^T + P_1 M_1 + Y \Delta M_2 + P_1 M_{d1} + Y \Delta M_{d2} + I_{r+\bar{q}}$$

$$\Pi_{12} = -M_1^T P_1 \delta - M_2^T \Delta^T Y^T \delta - M_{d1}^T P_1 \delta + P_1 \alpha_1 + Y \Delta \alpha_2 + P_1 \varsigma_1 + Y \Delta \varsigma_2 + P_1 \delta - M_{d2}^T \Delta^T Y^T \delta$$

$$\Pi_{13} = \Pi_{35} = 0_1$$

$$\Pi_{14} = \Pi_{36} = \Pi_{37} = \Pi_{57} = \bar{0}_{21}^T$$

$$\Pi_{22} = -\alpha_1^T P_1 \delta - \alpha_2^T \Delta^T Y^T \delta - \varsigma_1^T P_1 \delta - \varsigma_2^T \Delta^T Y^T \delta - \delta^T P_1 \alpha_1$$

$$-\delta^T Y \Delta \alpha_2 - \delta^T P_1 \varsigma_1 - \delta^T Y \Delta \varsigma_2 - 2P_3 + I_k$$

$$\Pi_{23} = \Pi_{45} = \bar{0}_{21}$$

$$\Pi_{24} = \Pi_{46} = \Pi_{47} = \Pi_{67} = 0_{k \times k}$$

$$\Pi_{15} = -\tau_m M_{d1}^T P_1 - \tau_m M_{d2}^T \Delta^T Y^T$$

$$\Pi_{16} = \tau_m M_{d1}^T P_1 \delta + \tau_m M_{d2}^T \Delta^T Y^T \delta$$

$$\Pi_{25} = -\tau_m \varsigma_1^T P_1 - \tau_m \varsigma_2^T \Delta^T Y^T$$

$$\Pi_{26} = \tau_m \varsigma_1^T P_1 \delta + \tau_m \varsigma_2^T \Delta^T Y^T \delta$$

$$\Pi_{17} = -P_1 \delta$$

$$\Pi_{27} = P_3$$

$$\Pi_{33} = \Pi_{55} = -(\tau_m^2 + 1)P_1$$

$$\Pi_{34} = \Pi_{56} = (\tau_m^2 + 1)P_1 \delta$$

$$\Pi_{44} = \Pi_{66} = -(\tau_m^2 + 1)P_3$$

$$\Pi_{77} = -2\gamma^2 I_k$$