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Revised Weighted Fuzzy C-Means and Fortified Weiszfeld Hybrid Method for Uncapacitated Multi-Facility Location Problems

Isik Guzelgoz, Department of Industrial Engineering, Istanbul University-Cerrahpasa, Turkey

Sakir Esnaf, Department of Industrial Engineering, Istanbul University-Cerrahpasa, Turkey

https://orcid.org/0000-0001-6261-3118

Tarik Kucukdeniz, Department of Industrial Engineering, Istanbul University-Cerrahpasa, Turkey

ABSTRACT

In this article, a hybrid method is proposed to solve the uncapacitated planar multi-facility location problems. The new hybrid method consists of a combination of the Revised Weighted Fuzzy C-Means (RWFCM) algorithm proposed by Esnaf and Küçükdeniz (2013) and the Fortified Weiszfeld algorithm developed by Drezner (2015). The cluster centers and the cluster assignments of the RWFCM are fed into the Fortified Weiszfeld Algorithm separately for each cluster and facility-customer allocations are determined. The proposed approach is benchmarked on sample datasets from the facility location literature. Results of the proposed hybrid method show that the newly proposed sequentially-run method achieves better results when compared against the benchmark methods. This paper is a pioneer study of the hybrid use of Revised Weighted Fuzzy C-Means and Fortified Weiszfeld algorithms.

KEYWORDS

Revised weighted fuzzy c-means algorithm, Uncapacitated multi-facility location problem, Weiszfeld and Fortified Weiszfeld algorithms

INTRODUCTION

In this paper, a hybrid algorithm to solve uncapacitated planar multi-facility location problems is proposed. The multi-facility location problem (MFLP) is the problem of placing a number of facilities to serve a group of customers such a way that the total cost is minimized. German engineer Alfred Weber gave the first example of MFLP problems in his famous book "On the Location of Industries" in 1909. In his book, Weber emphasized that the locations of industries are crucial for optimizing the costs (Weber, 1909).

Weber (1909) used and generalized the Fermat problem in order to minimize the costs. The Fermat problem is about finding a special point in a triangle so that the sum of its distances to the corners is minimized. The problem introduced first in a letter from Fermat to Torricelli in the 17^{th} century. Since then the same question arose and was answered in several forms. Weber (1909) reformulated the question by increasing the number of points from three to any number, and used the idea of "a weight" which is in fact firstly discussed in Steiner in 19^{th} century (Beck & Sabach, 2015).

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Hence, the problem is being now called the Fermat-Weber problem, or just the Weber Problem. Here, the aim is to find the optimum locations of facilities serving all the demand points whose locations are known, with condition that the sum of the distances from each demand point to the facility is minimized. In the original problem there is only one facility, and therefore it is also called "the single facility location problem". On the other hand, Multi-Facility Location Problem is a generalization of Weber problem. In MFLP there is more than one facility. (Iyigun & Ben-Israel, 2010)

The location problem and its solutions have a rich literature. Since the 1960s the problem is studied more thoroughly, and many different forms have been produced and many solution methods are proposed. Daskin (1995) gives a detailed classification which contains more than ten types of classes for the problem. Similarly, Sule (2001) gives 5 steps of classification. The diversity is due to the richness of the variety of constraints in different applications.

The solution methods first primarily depend on the number of facilities and classified as the Single Facility Location Problem (SFLP) and Multi-Facility Location Problem (MFLP). In this paper, Multi-Facility Location Problems are in focus. In many applications of MFLP, decomposing the problem into a single facility location is very common because these problems are NP-hard (Esnaf and Küçükdeniz, 2013).

Another important factor in solving MFLP is the structure of the facility and demand points: Continuous (planar), network or discrete structures. In this study the assumption is that the solution has a continuous structure, that is, facilities can be placed anywhere on the plane.

One another factor in constructing solutions for MFLP is the capacity of the facilities. The facilities can have limited or unlimited sources. The facilities have an unlimited capacity for the problems solved in this study.

Besides, other factors are static or dynamic structures of the facility, ownership of the facility as public or private, deterministic or probabilistic characteristics of the demand, the addition of facilities to the existing system or facility closure, and fixed and variable costs.

In this paper, it is focused on determining locations of uncapacitated facilities in a d-dimensional plane. The authors will suggest a combination of two previously known algorithms for the solution. The first algorithm is the revised weighted version of fuzzy c-means and the second is the empowered version of the Weiszfeld algorithm (Weiszfeld,1963) called Fortified Weiszfeld proposed by Drezner (2015).

The rest of the paper is organized as follows. The MFLP is described mathematically in the second section. In the third section, it is given a literature review of developments in recent decades. The fourth section contains the details of the revised weighted fuzzy c-means algorithm, Weiszfeld and Fortified Weiszfeld algorithms. The fifth section gives the details of the proposed algorithm. Afterward, the results of experiments on different data sets are given in the sixth section. The data sets used to make an objective comparison here are the same as Esnaf and Küçükdeniz (2013) in order to obtain a robust comparison. In the seventh and the last section, conclusions are discussed.

LITERATURE REVIEW

In order to solve MFLP, it is generally followed the method to divide the set of demand points into subsets and find supply center for each of them separately. Miehle (1958) is first to put forward this way of solving MFLP. Cooper (1963) gave the statement of Multi-Facility Weber problem formally (Rosing 1991). Cooper (1964) used this idea to propose an iterative heuristic method known as Alternate Location-Allocation (ALA) algorithm. The same way will be followed in this study: first decompose the set of demand points into sub-clusters and then tune the position of cluster centers so that optimal facility locations are obtained.

Since 1960s different composite methods are proposed to solve both uncapacitated and capacitated MFLPs. Kuenne and Soland (1972) applied a branch-and-bound algorithm. Logendran and Terrell (1988) introduced the stochastic uncapacitated facility location-allocation (UFLA) model. Taillard

(1996) proposed triple algorithm CLS, LOPT and DEC which can be used for the sets with a large number of demand points.

Levin and Ben-Israel (2004) suggested a hybrid method consists of the Nearest Center Reclassification Algorithm (known also as ALA) proposed by Cooper (1964) and Newton-Bracket method that is used for single facility problem.

Esnaf and Küçükdeniz (2009) proposed a hybrid method in which demand points are clustered using fuzzy c-means and afterward the center of each cluster is determined by center of gravity method. This method is used for uncapacitated continuous multi-facility location-allocation problem.

Esnaf and Küçükdeniz (2013) developed an algorithm named Revised Weighted Fuzzy C-Means to avoid using sequential computations and reducing CPU time.

Literature review reveals that the hybridization of fuzzy c-means and Weiszfeld type algorithms is a novel approach, which has not been investigated yet.

PROBLEM DEFINITION

Multi-facility location problem is an optimization problem with the cost function, which is defined below (Iyigün and Ben-Israel, 2010):

$$z = \sum_{i=1}^{c} \sum_{\overline{x_k} \in V_i} w_k d\left(\overline{x_k}, \overline{v_i}\right) \tag{1}$$

where,

$$\overline{x_k} = \left(x_k, y_k\right) = The\, location\, of\, customer\, k\, in\, a\, plane; k = 1,2,\ldots,n$$

$$w_k = The\, demand\, of\, customer\, k; w_k > 0 \;;\; k = 1,2,\ldots,n$$

$$\overline{v_i} = \left(p_i, q_i\right) = The\, location\, of\, center\, of\, facility\, i$$

$$V_i = Cluster\, of\, customers\, that\, is\, assigned\, to\, the\, ith\, facility$$

$$d\left(\overline{x_k}, \overline{v_i}\right) = distance\, between\, the\, facility\, i\, and\, customer\, k$$

Here, $d\left(\overline{x_k}, \overline{v_i}\right)$ is the Euclidean distance and calculated as follows:

$$d\left(\overline{x_k}, \overline{v_i}\right) = \sqrt{\left(x_k - p_i\right)^2 + \left(y_k - q_i\right)^2} \tag{2}$$

In this paper several assumptions are accepted for the MFLP (as in Lozano et al., 1998 and Esnaf and Küçükdeniz, 2013):

1- Facilities can be located anywhere on a plane, and their final locations are iteratively found.
2- Interactions between facilities are not allowed.
3- Each customer is only served by a single facility; in other words, customers may not split their demand between two or more facilities.
4- Transportation costs are assumed to be proportional to the Euclidean distance.
5- Each customer is assigned to its nearest facility.

6- Setup costs are omitted.
7- Customers spread over a continuous region of a plane and their locations and demands are fixed.

REVISED WEIGHTED FUZZY C-MEANS ALGORITHM (RWFCM) AND FORTIFIED WEISZFELD HYBRID ALGORITHM

In this paper the authors propose to use both RWFCM and Weiszfeld algorithms consecutively to get remarkable results for continuous multi-facility location problem. RWFCM first clusters the demand points and derives the cluster centers. After that, these cluster centers are fine-tuned by using Weiszfeld algorithm. In addition, a modified form of Weiszfeld algorithm called Fortified Weiszfeld by Drezner (2015) fine-tuned the cluster centers.

The details of these algorithms are given below.

Revised Weighted Fuzzy C-Means Algorithm (RWFCM)

The RWFCM (Esnaf, and Küçükdeniz 2013) is a specific form of the weighted fuzzy c-means algorithm proposed by Bezdek (1981), Tsekouras (2005) and Tsekouras et al. (2005). In this algorithm the weights of the demand points are given and accepted as constant during all iterations. In supply chain management context, demands of the customers are the weight factor used by the RWFCM algorithm.

In FCM and RWFCM same cost function is used:

$$z_p\left(U,v\right) = \sum_{k=1}^{n}\sum_{i=1}^{c} w_k \left(u_{ik}\right)^p a^k - v_i^2 \tag{3}$$

where c is the number of the final clusters, which coincides with the number of rules, $U = \left\{\left[u_{ik}\right], 1 \le i \le c\ , 1 \le k \le n\right\}$ is the partition matrix, $V = \left\{\left[v_i\right], 1 \le i \le c\right\}$ with $v_i \in \mathbb{R}^m$ is the vector of the final cluster prototypes, $a^k\left(1 \le k \le n\right)$ are the data to be clustered, $p \in \left(1, \infty\right)$ is a factor to adjust the membership degree weighting effect, and w_k is the weight of significance that is assigned to a^k.

The optimization problem is to minimize $z_p\left(U,v\right)$ under the following constraint:

$$\sum_{i=1}^{c} u_{ik} = 1\ ,\ \forall k \tag{4}$$

In FCM (Bezdek, 1981) the final prototypes and the respective membership functions that solve this constraint optimization problem are given by the following equations:

$$v_i = {\sum_{k=1}^{n} w_k \left(u_{ik}\right)^p a^k} \Big/ {\sum_{k=1}^{n} w_k \left(u_{ik}\right)^p} \tag{5}$$

and

$$u_{ik} = \frac{1}{\sum_{j=1}^{c} \left(\frac{a^k - v_i}{a^k - v_j} \right)^{\frac{2}{p-1}}} \tag{6}$$

In FCM, the weights are equal. On the other hand, RWFCM accepts different values for weights. The weights of points are seen as the demand amount of each point in RWFCM.

So, the steps of RWFCM are given as follows:

Step 1: c, the number of clusters, p, a value for the factor and initial values for facilities v_i are selectively determined.

Step 2: By using the formula (6) the membership values u_{ik} are calculated for $1 \leq i \leq c$,$1 \leq k \leq n$

Step 3: The new values of the facilities are obtained by the formula (5).

Step 4: The iterations are continued until $\max\left\{\left\| v_i - v_i^{new} \right\|_{err}\right\} < \varepsilon$ where ε is the error coefficient that is determined initially.

Weiszfeld Algorithm

Weiszfeld (1937) gave a different method to prove a theorem which had been given first by Sturm in 1884 for solving Fermat problem. In fact, he had given three different proofs and in the first one defined a sequence converges to an optimal solution of the problem.

From this sequence an algorithm can be deduced. Since the notion of algorithm is not familiar in those days, the paper had been ignored for a long time. First Miehle (1958), and later Kuhn and Kunne (1962) proposed similar methods and emphasized the convergence problem of the original algorithm. In 1963, Cooper also reinvented the method without knowing Weiszfeld's work, for the more general problem of multiple locations in plane (Beck and Sabach, 2015). Cooper did not provide a convergence analysis but mentioned that in his numerical tests, the method works very well in comparison to other methods. It seems that after 1963, researchers studying optimization and location problems were very well aware of the method, and Weiszfeld's original paper got its rightful credit (Beck and Sabach, 2015).

After 70's the researches focused on the convergence analysis and Kuhn (1973), Chandrasekaran and Tamir (1989), Brimberg (1995), Cánovas, et al. (2002) gave a detailed explanation on the convergence of the algorithm.

In Fermat-Weber problem objective function is given as (Beck and Sabach, 2015):

$$\min_{\vec{x}} \left\{ f\left(\vec{x}\right) = \sum_{i=1}^{n} w_i \, d\left(\vec{x}, a_i\right) \right\} \tag{7}$$

where $w_i > 0$,$i = 1,2,\ldots,m$ are given weights and the vectors $a_1, a_2, \ldots, a_m \in \wp \subset \mathbb{R}^d$ are given fixed points (demands). Here d is the Euclidean metric defined on $\mathbb{R}^{\boldsymbol{d}}$.

From Calculus, it is known that at the extremum point of a vector function, its gradient is 0. That is $\nabla f\left(\vec{\boldsymbol{x}}\right) = 0$ for the minimum/maximum point of $f\left(\vec{\boldsymbol{x}}\right)$. Here the gradient of the objective function is given as:

$$\nabla f\left(\vec{x}\right) = \sum_{i=1}^{m} w_i \frac{\vec{x} - a_i}{\left\| \vec{x} - a_i \right\|} \quad ,\vec{x} \notin \wp \tag{8}$$

Note that the gradient is only defined on points different from a_i 's. Weiszfeld's original result is (Beck and Sabach, 2015):

Theorem: Suppose that all a_i 's are collinear. Then,

a. $\min\left\{f\left(\vec{\boldsymbol{x}}\right)\right\}$ has a unique optimal solution
b. Let x^* be the optimal solution. If $x^* \notin \wp$, then $\nabla f\left(\overrightarrow{x^*}\right) = 0$

If $x^* = a_i$ for some $i \in \left\{1,2,\ldots,m\right\}$ then the following inequality holds:

$$\left\|\sum_{j=1\ ,j\neq i}^{m} w_j \frac{\overrightarrow{x^*} - \overrightarrow{a_j}}{\left\|\overrightarrow{x^*} - \overrightarrow{a_j}\right\|}\right\| \le w_i \tag{9}$$

This theorem gives

$$x^* = \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\left\|\vec{x} - a_i\right\|}} \sum_{i=1}^{m} \frac{w_i a_i}{\left\|\vec{x} - a_i\right\|} \tag{10}$$

By defining a new operator T as $T\left(x\right) := \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\left\|\vec{x} - a_i\right\|}} \sum_{i=1}^{m} \frac{w_i a_i}{\left\|\vec{x} - a_i\right\|}$ So, the equation can be written as follows:

$$T\left(x\right) = x^* \tag{11}$$

In other words, $\forall y \in \mathbb{R}^d \setminus \wp,$

$$y = T\left(y\right)\ \ if\, and\, only\, if\ \ \nabla f\left(y\right) = 0 \tag{12}$$

Hence, this connection gives us Weiszfeld iterative formula:
Initialization: $x_o \in \mathbb{R}^d \setminus \wp$
General Step: $\left(k = 0,1,\ldots..\right)$ $x_{k+1} = T\left(x_k\right)$
The algorithm can be written using more suitable notation as follows:

$$\overrightarrow{x_{k+1}} = \frac{\sum_{i=1}^{m}\left(\frac{w_i \overrightarrow{x_i}}{d\left(\overrightarrow{x_k}, \overrightarrow{x_i}\right)}\right)}{\sum_{i=1}^{m}\left(\frac{w_i}{d\left(\overrightarrow{x_k}, \overrightarrow{x_i}\right)}\right)} \tag{13}$$

Modified Weiszfeld Algorithm

As the number of demand points is raised, the convergence becomes slower and the algorithm may require thousands of iterations. Due to this fact, some researchers focused on accelerating the Weiszfeld's algorithm, in parallel to convergence studies.

Ostresh (1978) suggested accelerating the algorithm by using the operator as follows:

$$T_\lambda\left(y\right) = y + \lambda\left(T\left(y\right) - y\right) \tag{14}$$

Ostresh (1978) and Drezner (1992) showed that the modified method defined by $\overrightarrow{x_{k+1}} = T_\lambda\left(\overrightarrow{x_k}\right)$ is convergent for $\lambda \in \left[1,2\right]$. Chen (1984) also gives a similar step-size modification using radials which gives the result that the convergence is valid for λ with the condition $1 \leq \lambda \leq \frac{k}{k-1}$ for k-dimensional case. The use of λ =1.8 is recommended to provide the best empirical results (Drezner, 1992). In his formulation formula above can be notified as follows:

$$X'' = \frac{\sum_{i=1}^{m}\left(\frac{w_i a_i}{\left\|X' - a_i\right\|}\right)}{\sum_{i=1}^{m}\left(\frac{w_i}{\left\|X' - a_i\right\|}\right)} \tag{15}$$

where X' is the starting point, and X'' is the next point reached by the algorithm. In this approach new point is calculated as:

$$\bar{X} = X' + \lambda\left(X'' - X'\right) \tag{16}$$

The algorithm is continued with this new point $\bar{X}$.

Drezner (1995) suggested using Steffensen's method to accelerate the value of $\bar{X}$ linearly. In this method each component of new point is calculated separately as follows:

$$\hat{x} = x_0 - \frac{\left(x_1 - x_0\right)^2}{x_2 + x_0 - 2x_1} \tag{17}$$

where $x_1 = f\left(x_0\right)$,$x_2 = f\left(x_1\right)$.

Besides these works, there are also studies on generalization of Weiszfeld algorithm using l_p metrics (Aftab and Hartley, 2015), different structures of space or convexity of the region (Eckhardt, 1980).

Fortified Weiszfeld Algorithm

Drezner (2015) suggested a new form for the Weiszfeld algorithm called Fortified Weiszfeld Algorithm. It has been shown that the algorithm is much faster than the original Weiszfeld method, in just a few steps, it is possible to reach the same value instead of hundreds steps of the original algorithm.

"The fortified Weiszfeld algorithm consists of two components: approximating a paraboloid and checking very few demand points to find whether they are optimal" (Drezner, 2015). For approximating paraboloid, the second order approximation of two-variable function is used.

In calculus, the second order approximation of a two-variable function, $F\left(x,y\right)$, near point $\left(x_0,y_0\right)$ is defined as follows:

$$F\left(x,y\right) \approx F\left(x_0,y_0\right) + \left.\frac{\partial F}{\partial x}\right|_{\left(x_0,y_0\right)} \left|x-x_0\right| + \left.\frac{\partial F}{\partial y}\right|_{\left(x_0,y_0\right)} \left|y-y_0\right| + \frac{1}{2}\left.\frac{\partial^2 F}{\partial x^2}\right|_{\left(x_0,y_0\right)} \left(x-x_0\right)^2$$

$$+\frac{1}{2}\left.\frac{\partial^2 F}{\partial y^2}\right|_{\left(x_0,y_0\right)} \left(y-y_0\right)^2 + \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{\left(x_0,y_0\right)} \left|x-x_0\right|\left|y-y_0\right|$$

If the point is at the origin, that is $\left(x_0,y_0\right) = \left(0\ ,0\right)$, then:

$$F\left(x,y\right) \approx F\left(0,0\right) + \left.\frac{\partial F}{\partial x}\right|_{\left(0,0\right)} \left|x\right| + \left.\frac{\partial F}{\partial y}\right|_{\left(0,0\right)} \left|y\right| + \frac{1}{2}\left.\frac{\partial^2 F}{\partial x^2}\right|_{\left(0,0\right)} \left(x\right)^2 + \frac{1}{2}\left.\frac{\partial^2 F}{\partial y^2}\right|_{\left(0,0\right)} \left(y\right)^2 + \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{\left(0,0\right)} \left|x\right|\left|y\right|$$

So the approximation is a quadratic function $v_1 + v_2 x + v_3 y + \frac{1}{2} v_4 x^2 + \frac{1}{2} v_5 y^2 + v_6 xy = 0$. In order to find the unknown coefficients, v_i's, of this equation, Drezner (2015) uses eight equations $f_j = F\left(\theta_j,\psi_j\right) - F\left(0,0\right)$ where $\left(\theta_j,\psi_j\right)$ are the points $\left\{\left(1,1\right),\left(1,-1\right),\left(-1,1\right),\left(-1,-1\right),\left(0,1\right),\left(0,-1\right),\left(1,0\right),\left(-1,0\right)\right\}$.

The equations are written as $A\overline{v} = \overline{f}$ where A_{8x5} is the matrix given below:

$$A = \begin{bmatrix} 1 & +2 & 1 & +2 & +2 \\ 1 & -2 & 1 & +2 & -2 \\ 1 & -2 & 1 & -2 & +2 \\ 1 & +2 & 1 & -2 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

The residuals are $A\overline{v} = \overline{f}$ and the sum of squares of errors is the product vectors $\left(Av - f\right)^T \left(Av - f\right)$.

By differentiating the matrix product by v it is obtained: $2\left(Av - f\right)^T A = 0$.

Note that here $Av - f$ is a 8×1 matrix and left hand side is 1x5 vector. By product and transverse rules of matrices:

$$\left(Av\right)^T A - f^T A = 0$$

$$v^T A^T A = f^T A$$

$$A^T A v = A^T f$$

$$v = \left(A^T A\right)^{-1} A^T f \tag{17}$$

Here $\left(A^T A\right)^{-1} A^T$ can be calculated and following matrix is obtained:

$$\left(A^T A\right)^{-1} A^T = \left(\frac{1}{120}\right) \begin{bmatrix} 12 & 12 & 12 & 12 & -24 & -24 & 36 & 36 \\ 15 & -15 & -15 & 15 & 0 & 0 & 0 & 0 \\ 12 & 12 & 12 & 12 & 36 & 36 & -24 & 24 \\ 10 & 10 & -10 & -10 & 0 & 0 & 10 & -10 \\ 10 & -10 & 10 & -10 & 10 & -10 & 0 & 0 \end{bmatrix} \tag{18}$$

Then the coefficient vector $\vec{v}$ is obtained by the formulas $v_1 = \left(12f_1 + 12f_2 + 12f_3 + 12f_4 - 24f_5 - 24f_6 + 36f_7 + 36f_8\right)$, etc.

Since the quadratic function $v_1 + v_2 x + v_3 y + \frac{1}{2} v_4 x^2 + \frac{1}{2} v_5 y^2 + v_6 xy = 0$ represents a paraboloid, the minimum point $\left(\overline{x}, \overline{y}\right)$ is given by the formula:

$$\overline{x} = \frac{v_2 v_5 - v_3 v_4}{v_1 v_3 - v_2^2}, \overline{y} = \frac{v_2 v_4 - v_1 v_5}{v_1 v_3 - v_2^2} \tag{19}$$

Since the Weiszfeld algorithm stuck at the demand points, numerous iterations are needed to approach the solution point. To prevent such cases, the important thing is to know whether the solution is on a demand point. For this purpose, Love et al. (1988) propose that demand point i is optimal if and only if the following condition is satisfied:

$$w_i^2 \geq \left(\sum_{k \neq i = 1}^{n} w_k \frac{x_k - x_i}{d_{ik}} \right)^2 + \left(\sum_{k \neq i = 1}^{n} w_k \frac{y_k - y_i}{d_{ik}} \right)^2 \tag{20}$$

An indicator vector $U = \left[u_i\right]$ is maintained. If demand point i was not checked for optimality $u_i = 0$ otherwise $u_i = 1$. The vector U is set to all zeroes.

When demand point i is 'close' to X'' and $u_i = 0$ demand point i is checked for optimality by the formula above.

If demand point i is optimal, it is returned as the solution. Otherwise, set $u_i = 1$ and proceed.

Closeness can be defined in many ways. It is selected as 'close' the demand point closest to X'. Identifying such a demand point requires very little extra effort because the distances to all demand points are calculated for the Weiszfeld iteration (Drezner, 2015).

The algorithm consists of the following steps:

In the beginning, location accuracy $\varepsilon > 0$ and a starting solution X' are given.

(1) Set $u_i = 0$ for $i = 1,2,\ldots,n$.
(2) Perform one Weiszfeld iteration obtaining X''.
(3) Find the demand point k closest to X'. The distances from X' to all demand points are calculated when obtaining X''. If $u_k = 0$ then
    (a) Check whether demand point k is optimal by the optimality condition (20).
    (b) If demand point k is optimal, stop with demand point k as the solution.
    (c) Otherwise, set $u_k = 1$.
(4) Calculate $\Delta = \left\| X' - X'' \right\|_2$, the Euclidean distance between X' and X''.
(5) If $\Delta < \epsilon$ then stop with X'' as the solution.
(6) Calculate $f_j = F\left(x'' + \theta_j \Delta, y'' + \psi_j \Delta\right) - F\left(x'', y''\right)$ for j = 1, …, 8.
(7) Calculate $v = \left(A^T A\right)^{-1} A^T f$ and derive $\bar{X} = \left(\bar{x}, \bar{y}\right)$ using (17) and (19)
(8) Set $X' = X'' + \Delta \bar{X}$ and go to Step 2.

REVISED WEIGHTED FUZZY C-MEANS – WEISZFELD ALGORITHMS BASED HYBRID METHOD FOR UNCAPACITATED MULTI-FACILITY LOCATION PROBLEMS

The proposed method intends to find the optimal locations of facilities which will serve to demand points, like warehouse and distribution centers. The goal is to minimize the total transportation cost from the facilities to the demand points.

Demand points are assumed to be placed on a continuous plane with coordinates $a^k = \left(x_k, y_k\right)$. For each point, the demand is as assigned as the weight of the point and symbolized by w_k. The transportation cost from a facility to a demand point is represented as the distance between the locations of these two points for per quantity of demand. Here the positions of the demand points a^k

and values of demand quantities are known beforehand. The proposed method assumes that the capacity of each facility is unlimited as mentioned above.

The method will be applied in the following steps:

Step 1: c, the number of clusters, and initial values of cluster centers, here facilities, are selectively determined.

Step 2: After calculating the membership values, the new values of the cluster centers are determined by the RWFCM algorithm. Iterations of the RWFCM are repeated until transportation costs remain unchanged.

Step 3: For each of the clusters, the centers are calculated by using selected Weiszfeld Algorithm. Options are original Weiszfeld $(\lambda = 1)$. Modified Weiszfeld $(\lambda = 1.8)$.and Fortified Weiszfeld $(\lambda = 1, \lambda = 1.8)$.

Step 4: The demand points are classified again by using new clusters centers.

Step 5: The objective function is calculated and when difference between consecutive values is less than a predefined certain error value, the iteration stops. Otherwise go to step3.

When the iterations completed, the final positions of cluster centers will be determined, and each demand point is allocated to the nearest facility.

In the end, the total cost is calculated using the formula below:

$$TC = \sum_{k=1}^{n}\sum_{i=1}^{c} w_k d_{ik} C_{ik} \tag{21}$$

where;

TC: Total Cost

w_k: The demand of k th demand point.

d_{ik}: The Euclidean distance between k th demand point and i th facility.

C_{ik}: The transportation cost between k th demand point and i th facility for per unit and per demand quantity.

EXPERIMENTAL STUDY

The new algorithm is applied to well-known data sets, generally preferred in clustering based multi-facility location studies. The authors' purpose is to show that this method solves the uncapacitated continuous multi-facility location problem with a lower cost than the benchmark algorithms.

For this purpose, the data sets given by Esnaf and Küçükdeniz (2013) are chosen to show the performance of the RWFCM & Fortified Weiszfeld method. In this paper, particularly the same data sets were chosen to compare the results with the developed hybrid method. These data sets include customer number, *X* and *Y* coordinates and demands of each customer. The demand quantities are constant for all iterations of the algorithm.

Benchmarking methods are briefly explained as follows:

Fuzzy C-Means Algorithm (FCM): This is an algorithm for clustering the demand points using fuzzy membership values. Dunn (1973) developed the algorithm then it is improved by Bezdek (1981).

FCM&COG Algorithm: This is a hybrid method that uses the FCM algorithm and Center of Gravity method. First, the FCM determines the cluster centers and then the center of gravity method, fine-tunes the cluster centers of FCM, trying to minimize the total cost. It is proposed by Esnaf & Küçükdeniz (2009).

Particle Swarm Optimization (PSO) method: The PSO is developed by Kennedy and Eberhart (1995) and improved with the modifications suggested by Mezura-Montes and Coello Coello (2011). In this study the conditions of the PSO is considered completely same as in Esnaf and Küçükdeniz (2013).

Table 1 shows the total costs of benchmarking methods, RWFCM, and hybrid methods. Here in the hybrid methods, it is used Weiszfeld with λ=1, Modified Weiszfeld with λ=1.8 and Fortified Weiszfeld with λ=1 and λ=1.8 combined with the RWFCM.

To show the cost performance in percentage changes the following formula is used:

$$\Delta = \left(\frac{H - M}{H}\right) \times 100 \tag{22}$$

where H represents the objective function value, i.e. transportation cost, generated by benchmark methods (FCM, FCM&COG, PSO and RWFCM) for each data set, M is the transportation cost generated by the new hybrid methods for the corresponding data set.

Table 2 gives the percentage of cost differences of benchmark methods, given above, with the proposed hybrid method RWFCM & Weiszfeld using $\lambda = 1$.

RWFCM & Weiszfeld using λ=1 is 25.67%, 11.55%, 21.32%,17.4%, and 1.9% better than FCM, FCM&COG, PSO mean, PSO minimum, and RWFCM, respectively.

The following Table 3 gives the comparisons of the results obtained when it is applied the Modified Weiszfeld and Fortified Weiszfeld algorithms instead of Weiszfeld algorithm itself in the hybrid algorithm. It is clear that for these data sets, the difference between Modified and Fortified algorithms is minor.

Since the differences are minor, it is difficult to compare the results when using Modified or Fortified Weiszfeld algorithms in the hybrid method. The authors prepared Table 4 to show the differences between the costs of different versions of the proposed method and the RWFCM results in numbers. Therefore, the authors get slightly but better results than Drezner (1992) and Drezner (2015).

The differences between the costs of the RWFCM and the different versions of the RWFCM & Weiszfeld method are given in Table 4 below.

Since the values depend on the number of the data set, the effect of methods may be more clear on percentage values. The following Table 5 is prepared for clarification.

RWFCM & Fortified Weiszfeld algorithm with 1.8 lambda value attains the highest cost improvement ratio with 2.043%. RWFCM & Modified Weiszfeld algorithm follows the first with 2.042%, using the same lambda value. The results given above show that more cost savings can be achieved by using hybrid methods with higher lambda values, here is 1.8.

CONCLUSION

This paper suggests a hybrid solution consisting of two different methods for MFLP. The first method is the Revised Weighted Fuzzy C-Means and the second is a gradient-based Weiszfeld algorithm. Hybridization depends on the application of these algorithms consecutively. The results show that when the classical or modified versions of the Weiszfeld algorithm run after RWFCM give better cost function values for multi-facility location problems.

For the experimental study, the method is applied to the data sets used by Esnaf and Küçükdeniz (2013) to compare with the results of the RWFCM algorithm. These are ten different data sets with twenty-six different settings from Osman and Christofies (1994), Bongartz et al. (1994), Lorena and Pereira (2002) and Taillard (2003).

Table 1. The total costs of the FCM, FCM&COG, PSO and RWFCM methods are compared with the hybrid method RWFCM & Weiszfeld (λ=1), RWFCM & Modified Weiszfeld (λ=1.8) and RWFCM & Fortified Weiszfeld (λ=1, λ=1.8) algorithms

Datasets	Noof Demand Pts	=Noofclusters	FCM	FCM&COG	PSO(Mean, Minimum)	RWFCM	RWFCM & Weiszfeld (λ=1)	RWFCM & Modified Weiszfeld (λ=1.8)	RWFCM & Fortified Weiszfeld (λ=1)
Bongartz	287	10	13,632	9,373	8,510 7,882	7,627.66	7,488.59	7,488.59	7,488.59
Bongartz	287	20	11,163	6,652	6,356 5,878	4,622.84	4,424.98	4,424.93	4,424.97
SJC3a	300	25	1,896,044	1,590,720	1,698,809 1,589,846	1,435,737.81	1,403,309.56	1,403,309.56	1,403,309.56
SJC3a	300	40	1,463,476	1,175,349	1,384,343 1,344,344	1,035,707.47	1,010,510.26	1,010,510.75	1,010,510.43
SJC4a	402	25	2,848,104	2,520,080	2,727,118 2,617,243	2,374,352.00	2,339,707.29	2,339,698.20	2,339,706.09
SJC4a	402	40	2,387,161	1,977,533	2,337,425 2,180,875	1,840,615.86	1,792,436.56	1,792,436.26	1,792,436.45
SJC324	324	20	2,371,575	2,115,075	2,014,423 1,907,161	1,881,620.00	1,851,063.71	1,851,054.27	1,851,062.47
SJC324	324	40	1,582,704	1,250,084	1,447,463 1,384,221	1,163,306.29	1,137,948.95	1,137,947.33	1,137,948.75
SJC324	324	60	1,324,625	962,849	1,271,042 1,220,284	850,630.00	836,784,00	810,093.00	810,806.43
SJC500	500	20	5,027,370	4,827,415	4,571,051 4,422,492	4,269,917.30	4,177,700.42	4,177,698.87	4,177,700.37
SJC500	500	40	3,529,296	2,975,684	3,503,689 3,282,946	2,723,926.34	2,666,667.53	2,666,667.26	2,666,667.13
SJC500	500	60	2,918,302	2,400,974	3,118,047 2,984,051	2,091,680.61	2,043,055.44	2,043,052.09	2043055,226
SCJ708	708	20	6,920,864	6,599,399	6,606,900 6,338,001	6,017,948.00	5,952,604.78	5,952,604.78	5,952,604.78
SCJ708	708	40	5,042,815	4,226,576	4,993,834 4,874,041	3,998,800.65	3,935,825.20	3,935,810.24	3,935,823.22
SCJ708	708	60	3,983,870	3,331,938	4,315,360 4,078,910	2,989,036.00	2,934,786.96	2,934,584.14	2,934,483.99
SCJ818	818	20	9,867,565	8,733,462	9,103,742 8,893,566	7,624,484.86	7,533,597.20	7,533,597.20	7,533,597.20
SCJ818	818	40	6,293,308	5,543,320	6,828,894 6,523,902	5,250,380.22	5,136,107.75	5,136,094.41	5,136,106.95
SCJ818	818	60	5,358,010	4,147,632	5,877,620 5,565,993	3,873,383.00	3,797,735.30	3,797,470.25	3,797,428.96
CPmedcap2	1481	25	116,416	115,286	123,632 121,952	114,575.00	113,920.97	113,920.99	113,920.97
CPmedcap2	1481	40	89,777	88,913	98,147 95,700	88,556.55	87,894.16	87,894.16	87,894.16
Taillard	2863	25	968,287,764	748,719,913	852,102,000 780,562,000	656,694,127	645,895,203	645,894,892	645,895,185
Taillard	2863	50	637,134,339	497,671,334	642,804,992 602,791,270	452,711,742	443,107,998	443,107,275	443,107,922

The results of this new method were compared against the RWFCM, Fuzzy C-Means, Fuzzy C-Means with Center of Gravity, Particle Swarm Optimization algorithms. New algorithm gives better results with an average 1.899% than the RWFCM. The best improvement is on Bongartz data set with 20 clusters with 4.28% value. The lowest improvement for the cost function is 0.571% in the 25 cluster centers CPmedcap and 0.748% in the CPmedcap data set with 40 cluster centers.

The results of the basic version of the new method compared with FCM, FCM&COG and PSO methods are remarkable as the RWFCM. Average values for improvement are 25.67%,11.55%, 22.255%, and 21.32%(mean) with respect to FCM, FCM&COG and PSO methods consecutively.

Table 2. The percentage of cost differences of benchmark methods, given above, with the proposed hybrid method RWFCM & Weiszfeld using $\lambda = 1$.

DATA	Number of Demand Pts	Number of clusters	FCM	FCM & COG	PSO (Mean. Minimum)	RWFCM
Bongartz	287	10	45.07%	20.11%	12.01% 4.99%	1.82%
Bongartz	287	20	60.36%	33.48%	30.38% 24.72%	4.28%
SJC3a	300	25	25.99%	11.78%	17.39% 11.73%	2.26%
SJC3a	300	40	30.95%	14.03%	27.01% 24.83%	2.43%
SJC4a	402	25	17.85%	7.16%	14.21% 10.60%	1.46%
SJC4a	402	40	24.91%	9.36%	23.32% 17.81%	2.62%
SJC324	324	20	21.95%	12.48%	8.11% 2.94%	1.62%
SJC324	324	40	28.10%	8.97%	21.38% 17.79%	2.18%
SJC324	324	60	36.83%	13.09%	34.17% 31.43%	1.63%
SJC500	500	20	16.90%	13.46%	8.61% 5.54%	2.16%
SJC500	500	40	24.44%	10.39%	23.89% 18.77%	2.10%
SJC500	500	60	29.99%	14.91%	34.48% 31.53%	2.32%
SCJ708	708	20	13.99%	9.80%	9.90% 6.08%	1.09%
SCJ708	708	40	21.95%	6.88%	21.19% 19.25%	1.58%
SCJ708	708	60	26.33%	11.92%	31.99% 28.05%	1.82%
SCJ818	818	20	23.65%	13.74%	17.25% 15.29%	1.19%
SCJ818	818	40	18.39%	7.35%	24.79% 21.27%	2.18%
SCJ818	818	60	29.12%	8.44%	35.39% 31.77%	1.95%
CPmedcap	1481	25	2.14%	1.18%	7.86% 6.59%	0.58%
CPmedcap	1481	40	2.10%	1.15%	10.47% 8.16%	0.75%
Taillard	2863	25	33.29%	13.73%	24.20% 17.25%	1.64%
Taillard	2863	50	30.46%	10.96%	31.07% 26.49%	2.12%
Mean Value			25.67%	11.55%	21.32% 17.40%	1.90%

In the proposed method the Weiszfeld, Modified-Weiszfeld and Fortified-Weiszfeld algorithms were used. The results of each method slightly differed from each other. When compared total costs,

Table 3. The total costs of the RWFCM and hybrid methods the RWFCM & Weiszfeld (λ=1), RWFCM & Modified Weiszfeld (λ=1.8), and RWFCM & Fortified Weiszfeld (λ=1, and λ=1.8) algorithms. (λ=1.8 is suggested by Drezner,1992 and bold values show the lowest costs)

DATA	No of Demand Pts	No of clusters	RWFCM	RWFCM & Weiszfeld) (λ=1)	RWFCM & Modified Weiszfeld (λ=1,8)	RWFCM & Fortified Weiszfeld (λ=1)	RWFCM & Fortified Weiszfeld (λ=1,8)
Bongartz	287	10	7,627.6646	7,488.5927	7,488.5913	7,488.5913	7,488.5913
Bongartz	287	20	4,622.8436	4,424.9768	4,424.9298	4,424.9699	**4,424.9292**
SJC3a	300	25	1,435,737.8102	1,403,309.5616	1,403,309.5616	1,403,309.5616	1,403,309.5616
SJC3a	300	40	1,035,707.4684	1,010,510.2616	1,010,510.7522	1,010,510.4336	**1,010,508.7346**
SJC4a	402	25	2,374,352.00	2,339,707.2866	2,339,698.2010	2,339,706.0928	**2,339,698.1215**
SJC4a	402	40	1,840,615.86	1,792,436.5558	1,792,436.2624	1,792,436.4458	**1,792,436.2575**
SJC324	324	20	1,881,620.00	1,851,063.7147	1,851,054.2696	1,851,062.4741	**1,851,054.1880**
SJC324	324	40	1,163,306.29	1,137,948.9527	1,137,947.3265	1,137,948.7486	**1,137,947.2428**
SJC324	324	60	850,630.00	836,784.00	810,093.00	810,806.4273	**810,033.5966**
SJC500	500	20	4,269,917.30	4,177,700.4155	4,177,698.8724	4,177,700.3651	**4,177,698.8704**
SJC500	500	40	2,723,926.34	2,666,667.5252	2,666,667.2599	2,666,667.1335	**2,666,655.2172**
SJC500	500	60	2,091,680.61	2,043,055.4375	2,043,052.0925	2,043,055.2255	**2,043,051.91**
SCJ708	708	20	6,017,948.00	**5,952,604.7847**	**5,952,604.7847**	5,952,604.7848	5,952,604.7848
SCJ708	708	40	3,998,800.65	3,935,825.2001	3,935,810.2419	3,935,823.2190	**3,935,810.2383**
SCJ708	708	60	2,989,036.00	2,934,786.9623	2,934,584.1381	2,934,483.9863	**2,934,477.8598**
SCJ818	818	20	7,624,484.86	**7,533,597.2013**	**7,533,597.2013**	7,533,597.2016	7,533,597.2016
SCJ818	818	40	5,250,380.22	5,136,107.7487	5,136,094.4131	5,136,106.9498	**5,136,094.3597**
SCJ818	818	60	3,873,383.00	3,797,735.3049	3,797,470.2487	3,797,428.9593	**3,797,428.8265**
CPmedcap2	1481	25	114,575.00	**113,920.9719**	113,920.9900	113,920.9730	113,920.9907
CPmedcap2	1481	40	88,556.55	87,894.1588	87,894.1588	87,894.1588	87,894.1588
Taillard	2863	25	656,694,127	645,895,203	645,894,892	645,895,185	**645,894,891**
Taillard	2863	50	452,711,742	443,107,998	443,107,275	443,107,922	**443,107,264**

RWFCM & Weiszfeld algorithm (λ=1), RWFCM & Modified Weiszfeld algorithm (λ=1.8), RWFCM & Fortified Weiszfeld algorithm (λ=1), and RWFCM & Fortified Weiszfeld algorithm (λ=1.8) are, on average, 1.899%, 2.042%, 2.038%, and 2.043% better than the original RWFCM respectively. The contribution of hybrid methods at higher lambda values is more than small lambda values.

The results show that hybridization with Weiszfeld algorithm can improve the results obtained by a non-gradient based algorithm. So, for the future research, it can be focused on hybridization with different algorithms.

The modified Weiszfeld algorithm uses lambda with a value between 1 and 2 (Ostresh,1978). Drezner (1992) suggests 1.80 as an optimum value. This value can be examined in detail by using meta-heuristic optimization methods that accelerate the algorithm.

Finally, with this method, the capacitated multi-facility location problems can be solved.

Table 4. The differences between the costs of the RWFCM and the versions of the hybrid RWFCM & Weiszfeld method (bold values indicate the biggest cost differences)

DATA	No of Demand Pts	No of clusters	RWFCM & Weiszfeld (λ=1)	RWFCM & Modified Weiszfeld (λ=1.8)	RWFCM & Fortified Weiszfeld (λ=1)	RWFCM & Fortified Weiszfeld (λ=1.8)
Bongartz	287	10	139.0719	139.0733	139.0733	139.0733
Bongartz	287	20	197.8668	197.9138	197.8737	**197.9144**
SJC3a	300	25	32,428.2486	32,428.2486	32,428.2486	32,428.2486
SJC3a	300	40	25,197.2068	25,196.7162	25,197.0348	**25,198.7338**
SJC4a	402	25	34,644.7134	34,653.7990	34,645.9072	**34,653.8785**
SJC4a	402	40	48,179.3042	48,179.5976	48,179.4142	**48,179.6025**
SJC324	324	20	30,556.2853	30,565.7304	30,557.5259	**30,565.8120**
SJC324	324	40	25,357.3373	25,358.9635	25,357.5414	**25,359.0472**
SJC324	324	60	13,846.0000	40,537.0000	39,823.5727	**40,596.4034**
SJC500	500	20	92,216.8845	92,218.4276	92,216.9349	**92,218.4296**
SJC500	500	40	57,258.8148	57,259.0801	57,259.2065	**57,271.1228**
SJC500	500	60	48,625.1725	48,628.5175	48,625.3845	**48,628.7000**
SCJ708	708	20	**65,343.2153**	**65,343.2153**	65,343.2152	65,343.2152
SCJ708	708	40	62,975.4499	62,990.4081	62,977.4310	**62,990.4117**
SCJ708	708	60	54,249.0377	54,451.8619	54,552.0137	**54,558.1402**
SCJ818	818	20	**90,887.6587**	**90,887.6587**	90,887.6584	90,887.6584
SCJ818	818	40	114,272.4713	114,285.8069	114,273.2702	**114,285.8603**
SCJ818	818	60	75,647.6951	75,912.7513	75,954.0407	**75,954.1735**
CPmedcap2	1481	25	**654.0281**	654.0100	654.0270	654.0093
CPmedcap2	1481	40	662.3912	662.3912	662.3912	662.3912
Taillard	2863	25	10,798,924	10,799,235	10,798,942	**10,799,236**
Taillard	2863	50	9,603,744	**9,604,467**	9,603,740	9,604,466

Table 5. The percentages of performance of the methods in Table 4 with respect to the RWFCM

DATA	No of Demand Pts	No of clusters	RWFCM	RWFCM& Weiszfeld $\left(\lambda = 1\right)$	RWFCM& Modified Weiszfeld $\left(\lambda = 1.8\right)$	RWFCM& Fortified Weiszfeld $\left(\lambda = 1\right)$	RWFCM& Fortified Weiszfeld $\left(\lambda = 1,8\right)$
Bongartz	287	10	7,627.6646	1.823%	1.823%	1.823%	1.823%
Bongartz	287	20	4,622.8436	4.280%	4.281%	4.280%	4.281%
SJC3a	300	25	1,435,737.8102	2.259%	2.259%	2.259%	2.259%
SJC3a	300	40	1,035,707.4684	2.433%	2.433%	2.433%	2.433%
SJC4a	402	25	2,374,352.00	1.459%	1.46%	1.459%	1.46%
SJC4a	402	40	1,840,615.86	2.618%	2.618%	2.618%	2.618%
SJC324	324	20	1,881,620.00	1.624%	1.624%	1.624%	1.624%
SJC324	324	40	1,163,306.29	2.18%	2.18%	2.18%	2.18%
SJC324	324	60	850,630.00	1.628%	4.766%	4.682%	4.773%
SJC500	500	20	4,269,917.30	2.16%	2.16%	2.16%	2.16%
SJC500	500	40	2,723,926.34	2.102%	2.102%	2.102%	2.103%
SJC500	500	60	2,091,680.61	2.325%	2.325%	2.325%	2.325%
SCJ708	708	20	6,017,948.00	1.086%	1.086%	1.086%	1.086%
SCJ708	708	40	3,998,800.65	1.575%	1.575%	1.575%	1.575%
SCJ708	708	60	2,989,036.00	1.815%	1.822%	1.825%	1.825%
SCJ818	818	20	7,624,484.86	1.192%	1.192%	1.192%	1.192%
SCJ818	818	40	5,250,380.22	2.176%	2.177%	2.176%	2.177%
SCJ818	818	60	3,873,383.00	1.953%	1.96%	1.961%	1.961%
CPmedcap2	1481	25	114,575.00	0.571%	0.571%	0.571%	0.571%
CPmedcap2	1481	40	88,556.55	0.748%	0.748%	0.748%	0.748%
Taillard	2863	25	656,694,127	1.644%	1.644%	1.644%	1.644%
Taillard	2863	50	452,711,742	2.121%	2.122%	2.121%	2.122%
AVERAGE:				**1.899%**	**2.042%**	**2.038%**	**2.043%**

REFERENCES

Beck, A., & Sabach, S. (2015). Weiszfeld's Method: Old and New Results. *Journal of Optimization Theory and Applications*, *164*(1), 1–40. doi:10.1007/s10957-014-0586-7

Bezdek, J. C. (1981). *Pattern Recognition with Fuzzy Objective Function Algorithms*. New York: Plenum Press. doi:10.1007/978-1-4757-0450-1

Brimberg, J. (1995). The Fermat-Weber location problem revisited. *Mathematical Programming*, *71*(1), 71–76. doi:10.1007/BF01592245

Chandrasekaran, R., & Tamir, A. (1989). Open questions concerning Weiszfeld's algorithm for the Fermat-Weber location problem. *Mathematical Programming*, *44*(1-3), 293–295. doi:10.1007/BF01587094

Cánovas, L., Cañavate, R., & Marín, A. (2002). On the convergence of the Weiszfeld algorithm. *Mathematical Programming*, *93*, 327–330. doi:10.1007/s101070200297

Chen, R. (1984). Solution of location problems with radial cost functions. *Computers and Mathematics with Applications*, *10*(1), 87–94. doi:10.1016/0898-1221(84)90089-0

Cooper, L. (1963). Location-Allocation Problems. *Operations Research*, *11*(:3), 331–343. doi:10.1287/opre.11.3.331

Cooper, L. (1964). Heuristic Methods For Location-Allocation Problems. *SIAM Review*, *6*(1), 37–53. doi:10.1137/1006005

Daskin, M. S. (1995). *Network and discrete location: Models, algorithms, and applications*. New York: John Wiley & Sons, Inc. doi:10.1002/9781118032343

Drezner, Z. (1992). A note on the Weber location problem. *Annals of Operations Research*, *40*(1), 153–161. doi:10.1007/BF02060474

Drezner, Z. (1995). A note on accelerating the Weiszfeld procedure. Location Science, 3(4), 275-279.

Drezner, Z. (2015). The fortified Weiszfeld algorithm for solving the Weber problem. *IMA Journal of Management Mathematics*, *26*(1), 1–9. doi:10.1093/imaman/dpt019

Dunn, J. (1973). A Fuzzy Relative of the ISODATA Process and Its Use in Detecting Compact Well-Separated Clusters. *Journal of Cybernetics*, *3*(3), 32–57. doi:10.1080/01969727308546046

Eckhardt, U. (1980). Weber's Problem and Weiszfeld's Algorithm in General Spaces. *Journal Mathematical Programming*, *18*(1), 186–196. doi:10.1007/BF01588313

Esnaf, Ş., & Küçükdeniz, T. (2009). A Fuzzy Clustering-based Hybrid Method for a Multi-Facility Location Problem. *Journal of Intelligent Manufacturing*, *20*(2), 259–265. doi:10.1007/s10845-008-0233-y

Esnaf, Ş., & Küçükdeniz, T. (2013). Solving Uncapacitated Planar Multi-facility Location Problems by a Revised Weighted Fuzzy c-means Clustering Algorithm. *Journal of Multiple-Valued Logic & Soft Computing*, *21*, 147–164.

Iyigun, C., & Ben-Israel, A. (2010). A generalized Weiszfeld method for the multi-facility location problem. *Operations Research Letters*, *38*(3), 207–214. doi:10.1016/j.orl.2009.11.005

Kennedy, J., & Eberhart, R. (1995). Particle Swarm Optimization. In *Proceedings of the IEEE International Conference on Neural Networks*, Perth, Australia (pp. 1942–1945). doi:10.1109/ICNN.1995.488968

Kuenne, R. E., & Soland, R. M. (1972). Exact and approximate solutions to the multisource Weber problem. *Mathematical Programming*, *3*(3), 193–209. doi:10.1007/BF01584989

Kuhn, H. W., & Kuenne, R. E. (1962). An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics. *Journal of Regional Science*, *4*(2), 21–34. doi:10.1111/j.1467-9787.1962.tb00902.x

Kuhn, H. W. (1973). A note on Fermat's problem. *Mathematical Programming*, *4*(1), 98–107. doi:10.1007/BF01584648

Levin, Y., & Ben-Israel, A. (2004). A Heuristic Method For Large-scale Multi-facility Location Problems. *Computers & Operations Research*, *31*(2), 257–272. doi:10.1016/S0305-0548(02)00191-0

Logendran, R., & Terrell, M. P. (1988). Uncapacitated plant location-allocation problems with price sensitive stochastic demands. *Computers & Operations Research*, *15*(2), 189–198. doi:10.1016/0305-0548(88)90011-1

Love, R. F., Morris, J. G., & Wesolowsky, G. O. (1988). *Facilities Location: Models & Methods*. New York, NY: North Holland.

Lozano, S., Guerrero, F., Onieva, L., & Larraneta, J. (1998). Kohonen maps for solving a class of location-allocation problems. *European Journal of Operational Research*, *108*(1), 106–117.

Mezura-Montes, E., & Coello Coello, C. A. (2011). Constraint-handling in nature-inspired numerical optimization: Past, present and future. *Swarm and Evolutionary Computation*, *1*(4), 173–194. doi:10.1016/j.swevo.2011.10.001

Miehle, W. (1958). Link-length minimization in networks. *Operations Research*, *6*(2), 232–243. doi:10.1287/opre.6.2.232

Ostresh, L. M. Jr. (1978). On the convergence of a class of iterative methods for solving the Weber location problem. *Operations Research*, *26*(4), 597–609. doi:10.1287/opre.26.4.597

Rosing, K. E. (1992). An optimal method for solving the (generalized) Multi-Weber problem. *European Journal of Operational Research*, *58*(3), 414–426. doi:10.1016/0377-2217(92)90072-H

Sule, D. R. (2001). *Logistics of facility location and allocation*. New York: Marcel Dekker Inc. doi:10.1201/9780203910405

Tsekouras, G., Sarimveis, H., Kavakli, E., & Bafas, G. (2005). A hierarchical fuzzy-clustering approach to fuzzy modeling. *Fuzzy Sets and Systems*, *150*(2), 245–266. doi:10.1016/j.fss.2004.04.013

Tsekouras, G. E. (2005). On the use of the weighted fuzzy c-means in fuzzy modeling. *Advances in Engineering Software*, *36*(5), 287–300. doi:10.1016/j.advengsoft.2004.12.001

Weber, A. (1909). ÜBer Den Standort Der Industrien, 1. Teil: Reine Theorie Des Standortes. Tübingen. English Translation: On the Location of Industries. Chicago, IL: University of Chicago Press.

Weiszfeld, E. (1936). Sur le point pour lequel la somme des distances de n points donnes est minimum. *Tohoku Mathematical Journal*, *43*, 355–386.

M. Işik Guzelgoz is a Ph.D. student of the Department of Industrial Engineering at Istanbul University-Cerrahpasa. He graduated from Bogazici University, Department of Mathematics in 1993. He received his M.Sc. from the Department of Mathematics at Middle East Technical University. He worked as an IT analyst at Garanti Bank, Istanbul for ten years. He has been working as IT analyst at Gunes Sigorta, Istanbul since 2009. He is now researching at facility location and optimization theory.

Sakir Esnaf is a Professor of Department of Industrial Engineering at Istanbul University-Cerrahpasa. He graduated from Istanbul Technical University, Department of Industrial Engineering in 1986. He received his M.Sc. and Ph.D. degrees in Production Management from Istanbul University. He worked as a postdoctoral research fellow at the Intelligent Fuzzy Systems Laboratory, Department of Mechanical and Industrial Engineering at the University of Toronto, Canada. His current research topics are facility location, fuzzy clustering, metaheuristics, big data analytics, and revenue management. He has several national and international papers published.

Tarik Kucukdeniz is an Associate Professor of Industrial Engineering in the Engineering Faculty of Istanbul University-Cerrahpasa. He received his M.Sc. and Ph.D. degrees from the Industrial Engineering Department of Istanbul University. His current research topics include optimization methods in supply chain problems and machine learning algorithms for time series prediction and regression.