INTRODUCTION

Graph theory has numerous application to problems in systems analysis, operations research, economics, and transportation. However, in many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic.

Hypergraphs (Berge, 1989) are the generalization of graphs in case of set of multiarity relations. It means the expansion of graph models for the modeling complex systems. In case of modelling systems with fuzzy binary and multiarity relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realise formal optimisation and logical procedures.

MAIN DEFINITIONS OF FUZZY GRAPHS AND HYPERGRAPHS

This article presents the main notations of fuzzy graphs and fuzzy hypergraphs, invariants of fuzzy graphs and hypergraphs.

Fuzzy Graph

Let a fuzzy direct graph \( \tilde{G} = (X, \tilde{U}) \) is given, where \( X \) is a set of vertices, \( \tilde{U} = \{ \mu_U(x_i, x_j) \mid (x_i, x_j) \in X^2 \} \) is a fuzzy set of edges with the membership function \( \mu_U : X^2 \rightarrow [0, 1] \) (Kaufmann, 1977).

Example 1. Let fuzzy graph \( \tilde{G} \) has \( X = \{ x_1, x_2, x_3, x_4 \} \), and \( \tilde{U} = \{ <0.5/\{x_1,x_2\}>, <0.6/\{x_1,x_3\}>, <0.3/\{x_1,x_4\}>, <0.2/\{x_3,x_4\}>, <1/\{x_4,x_2\}> \} \). It is presented in figure 1.

The fuzzy graph \( \tilde{G} \) may present a fuzzy dependence relation between objects \( x_i, x_j, x_1, \) end \( x_4 \). If the object \( x_i \) fuzzy depends from the object \( x_j \), then there is direct edge \( (x_i, x_j) \) with membership function \( \mu_{E_i}(x_i, x_j) \).

If a fuzzy relation, presented by fuzzy graph \( \tilde{G} \), is symmetrical, we have the fuzzy nondirect graph.
Fuzzy Graphs and Fuzzy Hypergraphs

A fuzzy graph $\tilde{G} = (X, \tilde{U})$ is convenient for representing as fuzzy adjacent matrix $[r_{ij}]_{n \times n}$, where $r_{ij} = \mu_{ij}(x_i, x_j)$. So, the fuzzy graph, presented in figure 1, may be consider by adjacent matrix:

$$
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  x_1 & 0 & 0.5 & 0 & 0 \\
  x_2 & 0 & 0 & 0.6 & 0 \\
  x_3 & 0 & 0 & 0 & 0.3 \\
  x_4 & 0.2 & 1 & 0 & 0 \\
\end{array}
$$

The fuzzy graph $\tilde{H} = (X', \tilde{U}')$ is called a fuzzy subgraph (Monderson & Nair, 2000) of $\tilde{G} = (X, \tilde{U})$ if $X' \subseteq X$ and $\tilde{U}' \subseteq \tilde{U}$.

Fuzzy directed path (Bershtein & Bozhenyuk, 2005) $\tilde{L}(x_i, x_m)$ of graph $\tilde{G} = (X, \tilde{U})$ is called the sequence of fuzzy directed edges from vertex $x_i$ to vertex $x_m$:

$$\tilde{L}(x_i, x_m) = \langle \mu_{ij}(x_i, x_j)(x_j, x_k)(x_k, x_l)\rangle.$$

Conjunctive strength of path $\mu(\tilde{L}(x_i, x_m))$ is defined as:

$$\mu(\tilde{L}(x_i, x_m)) = \mu_{ij}(x_i, x_j) \& \mu_{ik}(x_k, x_l) \& \ldots$$

Fuzzy directed path $\tilde{L}(x_i, x_m)$ is called simple path between vertices $x_i$ and $x_m$ if its part is not a path between the same vertices.

If a number of vertices $n \geq 3$ and $x_i = x_m$, then the path is called a cycle.

Obviously, what is it definition coincides with the same definition for nonfuzzy graphs.

Vertex $y$ is called fuzzy accessible from vertex $x$ in the graph $\tilde{G} = (X, \tilde{U})$ if exists a fuzzy directed path from vertex $x$ to vertex $y$.

The accessible degree of vertex $y$ from vertex $x$, $(x \neq y)$ is defined by expression:

$$\gamma(x,y) = \max_{\alpha} (\mu(\tilde{L}_\alpha(x,y)), \alpha = 1,2,\ldots,p),$$

where $p$ - number of various simple directed paths from vertex $x$ to vertex $y$.

A subset of vertices $X'$ is called a fuzzy independent vertex set (Bershtein & Bozhenuk, 2001) with the degree of independence

$$\alpha (X') = 1 - \max_{\gamma \in X'} \mu_{ij}(x_i, x_j).$$

A subset of vertices $X' \subseteq X$ of graph $\tilde{G}$ is called a maximal fuzzy independent vertex set with the degree of independence $\alpha (X')$, if the condition $\alpha (X'') < \alpha (X')$ is true for any $X'' \subset X'$.

Let a set $\tau = \{ X_{i_1}, X_{i_2}, \ldots, X_{i_p} \}$ be given where $X_{i_k}$ is a fuzzy independent $k$-vertex set with the degree of independent $\alpha_{i_k}$. We define as

$$\alpha_k^{\text{max}} = \max \{ \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} \}.$$

The value $\alpha_k^{\text{max}}$ means that fuzzy graph $\tilde{G}$ includes $k$-vertex subgraph with the degree of independent $\alpha_k^{\text{max}}$ and doesn’t include $k$-vertex subgraph with the degree of independence more than $\alpha_k^{\text{max}}$.

A fuzzy set

$$\Psi_x = \{ < \frac{\alpha_k^{\text{max}}}{1}, < \frac{\alpha_k^{\text{max}}}{2}, \ldots, < \frac{\alpha_k^{\text{max}}}{n} > \}$$

is called a fuzzy independent set of fuzzy graph $\tilde{G}$.

Fuzzy graph $\tilde{G}$, presented in figure 1, has seven maximum fuzzy independent vertex sets:

$$\Psi_1 = \{ x_2 \}, \quad \Psi_2 = \{ x_4 \}, \quad \Psi_3 = \{ x_1, x_3 \}$$

with the degree of independence $1$; $\Psi_4 = \{ x_1, x_4 \}$ with the degree of independence $0.8$; $\Psi_5 = \{ x_1, x_3, x_4 \}$ with