Chapter IV
The Conception of the Sub–Pixel Efficacy Region

INTRODUCTION

This chapter presents the mathematical deduction of the Sub-pixel Efficacy Region (SRE) for the case of the trivariate liner interpolation function. The questions that the reader may have at this point are: (i) what is the need of defining the Sub-pixel efficacy Region once again? And (ii) why the Sub-pixel Efficacy Region is to be calculated on the basis of the Intensity-Curvature Functional?

To reconnect with the study undertaken in Chapters II and III it is due to recall that the SRE is only an intuition before the beginning of this chapter. Therefore, the answer to the first question is immediate to the writer to the extent that the SRE mathematically characterized here consists of a conceptual refinement to what already anticipated through the intuition presented in Chapter II. This is necessary in order to move the first step towards the determination of the SRE-based interpolation functions and this shall be seen throughout the rest of the book. Nevertheless, this is also consequential to the process of deduction that started from the intuition.

In this chapter, the Intensity-Curvature Functional is associated with the concept of metric employed to measure the intensity-curvature content of the interpolation function.

Nevertheless, Chapter III has presented the conception of the Intensity-Curvature Functional and concluded that such functional is conceived and demonstrated to be the measure of change of the energy of the signal (image) that the interpolation function determines. The answer to the second question is therefore the following: to calculate the SRE on the basis of the Intensity-Curvature Functional is conceptually equivalent to a process that finds the intra-pixel locations where the change of image energy determined through the model interpolation function is either minimal or maximal. This answer naturally descends from the fact that the SRE is calculated from the solution of the polynomial system of equations consisting of the three first order partial derivatives of the Intensity-Curvature Functional.

SUB-PIXEL EFFICACY REGION

The Sub-Pixel Efficacy Region is derived through the study of the Intensity-Curvature Functional, particularly through the process of finding the extremes of $\Delta E$. Based on the results of Chapter III, the
The Conception of the Sub-Pixel Efficacy Region

Intensity-Curvature Functional is herein calculated:

\[ \Delta E(x, y, z) = E_o / E_{IN} \]  

\[ E_{IN} = E_{IN} (x, y, z) = E_{IN} (\Psi_{xy}) + E_{IN} (\Psi_{zx}) + E_{IN} (\Psi_{yz}) = \]

\[ \iiint h(x, y, z) \left( \partial^2 (h(x,y,z)) / \partial x \partial y \right) dx dy dz + \iiint h(x, y, z) \left( \partial^2 (h(x,y,z)) / \partial z \partial x \right) dx dy dz + \iiint h(x, y, z) \left( \partial^2 (h(x,y,z)) / \partial y \partial z \right) dx dy dz \]  

\[ E_o = E_o (x, y, z) = E_o (\Psi_{xy}) + E_o (\Psi_{zx}) + E_o (\Psi_{yz}) = \]

\[ \iiint f(0, 0, 0) \left( \partial^2 (h(x,y,z)) / \partial x \partial y + \partial^2 (h(x,y,z)) / \partial y \partial z + \partial^2 (h(x,y,z)) / \partial x \partial z \right) (0, 0, 0) dx dy dz \]

It is due to clarify that above integrals are calculated from (0, 0, 0) (the origin of the coordinate system of the voxel, assumed at the location of the node of intensity f(0, 0, 0)) to anywhere (x, y, z) in the voxel.

For the rest of the interpolation functions treated in the book, the integrals of the intensity-curvature terms are calculated from the origin of the coordinate system in between two nodes (1D) or the pixel (2D) respectively, to anywhere within the intra-nodal distance or the pixel.

The calculation shown in equations (2) and (3) is different from the one previously seen in Chapter III, where the integrals where calculated from (-x/2, -y/2, -z/2) to (x/2, y/2, z/2) such to quantify the energy level of the signal (image) obtained before and after interpolation. In this chapter the concept of energy is temporarily subsidized by the concept of Intensity-Curvature content embedded into the model interpolation function at a given re-sampling location (x, y, z) in the voxel.

From this concept it is derived the measure of intensity-curvature of the interpolation function, which, herein and in the rest of the book will be called: the Intensity-Curvature Functional. Hereto is presented the calculation of the integrals and also for what concerns the deduction of the Sub-pixel Efficacy Region is presented the math process that derives the extremes of the ratio expressed by equation (1). Keeping in mind that \( (\partial^2 (h(x, y, z)) / \partial x \partial y) = - \theta_{xy} \omega_x + z \omega_y \) is a function of the variable z only, and that \( (\partial^3 (h(x, y, z)) / \partial x \partial y \partial z) = \omega_z \) and also by the benefits of Lemma III of Chapter III, let us proceed to write:

\[ E_{IN} (\Psi_{xy}) = \iiint h(x, y, z) \left( \partial^2 (h(x,y,z)) / \partial x \partial y \right) dx dy dz = \]

\[ \iiint H(x, y, z) \left( \partial^2 (h(x,y,z)) / \partial x \partial y \right) dy dz = \]