A Unified Approach To Fractal Dimensions

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ABSTRACT

Many scientific papers treat the diversity of fractal dimensions as mere variations on either the same theme or a single definition. There is a need for a unified approach to fractal dimensions for there are fundamental differences between their definitions. This paper presents a new description of three essential classes of fractal dimensions based on: (1) morphology, (2) entropy, and (3) transforms, all unified through the generalized-entropy-based Rényi fractal dimension spectrum. It discusses practical algorithms for computing 15 different fractal dimensions representing the classes. Although the individual dimensions have already been described in the literature, the unified approach presented in this paper is unique in terms of (1) its progressive development of the fractal dimension concept, (2) similarity in the definitions and expressions, (3) analysis of the relation between the dimensions, and (4) their taxonomy. As a result, a number of new observations have been made, and new applications discovered. Of particular interest are behavioral processes (such as dishabituation), irreversible and birth-death growth phenomena (e.g., diffusion-limited aggregates (DLAs), dielectric discharges, and cellular automata), as well as dynamical non-stationary transient processes (such as speech and transients in radio transmitters), multi-fractal optimization of image compression using learned vector quantization with Kohonen’s self-organizing feature maps (SOFMs), and multi-fractal-based signal denoising.

Keywords: fractal dimensions; monofractals and multi-fractals; relationship between fractal dimensions; unified framework for fractal dimensions

INTRODUCTION

This article is concerned with measuring the quality of various multimedia materials used in perception, cognition, and evolutionary learning processes. The multimedia materials may include temporal signals such as sound, speech, music, biomedical and telemetry signals, as well as spatial signals such as still images, and spatio-temporal signals such as animation and video. A comprehensive review of the scope of multimedia storage and transmission, as well as quality metrics is presented by Kinsner (2002). Most of such original materials are altered (compressed or enhanced) either to fit the available storage
or bandwidth during their transmission, or to enhance perception of the materials. Since the signals may also be contaminated by noise during different stages of their processing and transmission, various denoising techniques must be used to minimize the noise, without affecting the signal itself (Kinsner, 2002). Different classes of colored and fractal noise are described by Kinsner (1994c). A review of approaches to distinguish broadband signals and noise from chaos was provided by Kinsner (2003). The multimedia compression is often lossy in that the signals are altered with respect not only to their redundancy, but also to their perceptual and cognitive relevancy. Since the signals are presented to humans (rather than machines), cognitive processes must be considered in the development of suitable quality metrics. Energy-based metrics are not suitable for such cognitive processes. A very fundamental class of metrics based on entropy was described by Kinsner (2004), with a discussion on its usefulness and limitations in the area of cognitive informatics (CI) as defined in Wang (2002) and Wang and Kinsner (2006) and autonomous computing (Kinsner, Potter, & Faghfouri, 2005; Wang, 2007). This paper is an extension of the single-scale, entropy-based metrics to multi-scale metrics through fractal dimensions. Many experimental results obtained by the author and his collaborators indicate that quality metrics based on fractal dimensions appear to be most suited for perception. Further research on their suitability for cognition is being conducted.

A topological dimension is by definition a non-negative integer 0, 1, 2, … . The dimension of a general abstract vector space is the number of linearly independent vectors required for a basis, and is \( n \) for \( \mathbb{R}^n \). An orthonormal basis is by definition a basis which is an orthonormal set. The space-time we live in is often characterized by four Euclidean integer dimensions.

At the end of the 19th century, it was believed that one could define the dimension of a space as the number of continuous parameters required for describing it. However, with the introduction of space-filling curves by Peano, Hilbert, Minkowski, Sierpinski and others, as well as the discovery of continuous but nowhere differentiable curves by Weierstrass, and dusts by Cantor and Julia, the notion of the integer dimension had to be refined. Consequently, a real number was then introduced as a measure of the degree (index) of space filling (meandering, roughness, brokenness, or irregularity). Today, chaos in a dynamical system is characterized by such numbers as a measure of the corresponding complexity of a strange attractor of the system. In 1928, Bouligand called the space-filling index the Cantor-Minkowski order. It was also known as fractional dimension (Besicovitch in the 1930s), logarithmic density, and even capacity (Frostman) and KS entropy (after A.N. Kolmogorov 1958 and Sinai 1959). In the 1960s, Mandelbrot reformulated the number in terms of the Hausdorff-Besicovitch fractal dimension to acknowledge the fundamental contribution of Hausdorff in 1919 in establishing a measure of a set through successive coverings of the set by volume elements (abbreviated to vels), and the refinement of the idea by Besicovitch 10 years later. In 1979, Mandelbrot defined a fractal as “a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension” (Mandelbrot, 1982). Today, we also recognize that some (fat) fractals may have integer dimensions. Since the dimension plays such an important role in fractals, it has been described in numerous books and other sources such as

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