Chapter III

Variational Problems in Image Segmentation and $\Gamma$-Convergence Methods

Giovanni Bellettini, University of Roma, Italy
Riccardo March, Italian National Research Council, Italy

ABSTRACT

Variational models for image segmentation aim to recover a piecewise smooth approximation of a given input image together with a discontinuity set which represents the boundaries of the segmentation. In particular, the variational method introduced by Mumford and Shah includes the length of the discontinuity boundaries in the energy. Because of the presence of such a geometric term, the minimization of the corresponding functional is a difficult numerical problem. We consider a mathematical framework for the Mumford-Shah functional and we discuss the computational issue. We suggest the use of the $\Gamma$-convergence theory to approximate the functional by elliptic functionals which are convenient for the purpose of numerical computation. We then discuss the design of an iterative numerical scheme for image segmentation based on the $\Gamma$-convergent approximation. The relation between the Mumford-Shah model and the Perona-Malik equation will be also discussed.

INTRODUCTION

The segmentation problem in computer vision consists in decomposing an image into regions which correspond to meaningful parts of objects in a visual scene. The image intensity has to be as uniform as possible inside each region, while sharp transitions take place across the boundaries. Every piece of boundary is an intensity edge and the related problem of edge detection looks for the location of the sharp transitions in intensity. Edge
detection requires a further linking process of the edges into global curves to achieve segmentation. The variational approach to image segmentation unifies edge detection and linking into a single energy minimization method.

Edges are considered as the locations where the modulus of the gradient of image intensity is locally maximum, so that edge detection requires the evaluation of derivatives of this intensity. Torre and Poggio (1986) showed that, because of the presence of noise, the numerical differentiation of an image is an ill-posed mathematical problem. Numerical stability of image differentiation, then, requires a previous regularizing operation that can be attained by means of convolution of the image intensity with a suitable kernel. Marr and Hildreth (1980) proposed the convolution with a Gaussian function in their theory of edge detection, and a rigorous justification of Gaussian filtering before differentiation is given by Torre and Poggio (1986). Morel and Solimini (1995) pointed out that there exists an energy functional associated with the Marr and Hildreth theory. This follows from the observation that the convolution of the image intensity with a Gaussian function is equivalent to the solution of the heat equation with the image as initial datum:

\[
\frac{\partial u}{\partial t} = \Delta u, \quad u(x,0) = g(x)
\]

where \( g(x) \) denotes the image intensity. The solution of the heat equation in the whole space is given by:

\[
u(x,t) = (G_\sigma * g)(x), \quad G_\sigma(x) = \frac{1}{4\pi\sigma} \exp\left(-\frac{x^2}{4\sigma}\right)
\]

The heat equation is also the gradient flow of the Dirichlet functional:

\[
E(u) = (1/2) \int_\Omega |\nabla u|^2 \, dx
\]

with Neumann boundary conditions, where \( \Omega \) is the image domain. The heat equation requires the choice of a stopping time which corresponds to the parameter \( \sigma \) of the Gaussian function \( G_\sigma \). Such a parameter yields the spatial scale at which the edges have to be detected. A scale parameter can also be directly introduced in the differential equation by forcing the solution \( u \) to remain close to \( g \):

\[
\frac{\partial u}{\partial t} = \Delta u - \mu(u - g)
\]

Now this partial differential equation is the gradient flow of the functional:

\[
E(u) = (\mu / 2) \int_\Omega (u - g)^2 \, dx + (1/2) \int_\Omega |\nabla u|^2 \, dx
\]  

(1)