Hölder’s Inequality and Related Inequalities in Probability

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ABSTRACT

In this paper, the author examines Hölder’s inequality and related inequalities in probability. The paper establishes new inequalities in probability that generalize previous research in this area. The author places Beckenbach’s (1950) inequality in probability, from which inequalities are deduced that are similar to Brown’s (2006) inequality along with Olkin and Shepp (2006).

Keywords: Beckenbach’s Inequality, Hölder’s Inequality, Inequalities, Probability, Proof

1. INTRODUCTION

Yeh, Yeh, and Chan (2008) link some equivalent probability inequalities in a common probability space, such as Hölder, Minkowski, Radon, Cauchy, and so on. In this paper, we will establish some new inequalities in probability which generalize some inequalities (Sun, 1997; Wan, Su, & Wang, 1967; Wang & Wang, 1987; Yeh, Yeh, & Chan, 2008). We also establish Beckenbach’s (1950) inequality in probability, from which we deduce some inequalities which look like Brown’s (2006) inequality along with Olkin and Shepp (2006) and related results (Beckenbach & Bellman, 1984; Casella & Berger, 2002; Danskin, 1952; Dresher, 1953; Gurland, 1968; Hardy, Littlewood, & Polya, 1952; Kendall & Stuart; Loeve, 1998; Marshall & Olkin, 1979; Mullen, 1967; Persson, 1990; Sclove, Simons, & Ryzin, 1967; Yang & Zhen, 2004).

For convenience, throughout this paper, we let \( n \) be a positive integer and define

\[
E_p X = \begin{cases} 
(EX^p)^{1/p}, & p \neq 0 \\
\exp(E \ln X), & p = 0,
\end{cases}
\]

where \( EX \) denote the expected value of a nonnegative random variable \( X \). And we consider only the random variables which have finite expected values.

To establish our results, we need the following two lemmas: Lemma 1 (Yeh, Yeh, & Chang, 2008) and Lemma 2 due to Radon (Hardy, Littlewood, & Polya, 1952).

Lemma 1.

Let \( X \) and \( Y \) be nonnegative random variables on a common probability space. Then the following inequalities are equivalent:
(a) \( EX^h Y^k \leq (EX)^h (EY)^k \) if \( h + k = 1 \) with \( h > 0 \) and \( k > 0 \);
(b) \( EX^h Y^k \leq (EX)^h (EY)^k \) if \( h + k \leq 1 \) with \( h > 0 \) and \( k > 0 \);
(c) \( EX^h Y^k \geq (EX)^h (EY)^k \) if \( h + k = 1 \) with \( hk < 0 \);
(d) \( EX^h Y^k \geq (EX)^h (EY)^k \) if \( h + k \geq 1 \) with \( hk < 0 \);
(e) \( EX^p \geq (EX)^p \) if \( p \geq 1 \) or \( p \leq 0 \),
(f) \( EX^p \leq (EX)^p \) if \( 0 < p < 1 \);

2. Hölder’s Inequality

**Theorem 2.1.** Let \( X_1, X_2, \ldots, X_n \) be nonnegative random variables on a common probability space. Then the following two results hold:

\[
(R_1) \quad E(X_1^{q_1} X_2^{q_2} \cdots X_n^{q_n}) \leq (EX_1)^{q_1} (EX_2)^{q_2} \cdots (EX_n)^{q_n}
\]

if \( q_1, q_2, \ldots, q_n \in (0,1) \) with \( \sum_{i=1}^n q_i = 1 \);

\[
(R_2) \quad E(X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}) \leq (EX_1)^{\frac{r_1}{r}} (EX_2)^{\frac{r_2}{r}} \cdots (EX_n)^{\frac{r_n}{r}}
\]

if \( r_1, r_2, \ldots, r_n \in (0,1) \) with \( r = \sum_{i=1}^n r_i \). In particular, if \( r \leq 1 \), then

\[
(R_2') \quad E(X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n}) \leq (EX_1)^{\frac{1}{r}} (EX_2)^{\frac{1}{r}} \cdots (EX_n)^{\frac{1}{r}} .
\]

**Proof.** Case \( R_1 \): Inequality \( R_1 \) can be proved by mathematical induction using \( (a) \) of Lemma 1. But for the sake of completeness, we give its proof here. It follows from \( (a) \) of Lemma 1 that \( R_1 \) holds for \( n = 2 \). Suppose that \( R_1 \) holds for \( n = k \).

Thus, for \( n = k + 1 \), let \( q_1 + q_2 + \cdots + q_k = q \).

Then \( \sum_{i=1}^k \frac{q_i}{q} = 1 \) and \( q + q_{k+1} = 1 \). Hence:

\[
E(X_1^{q_1} X_2^{q_2} \cdots X_k^{q_k} X_{k+1}^{q_{k+1}}) = E((X_1^{q_1} X_2^{q_2} \cdots X_k^{q_k})^{q_{k+1}})
\]

\[
\leq E(X_1^{q_1} X_2^{q_2} \cdots X_k^{q_k})^{q_{k+1}} E(X_{k+1}^{q_{k+1}})
\]

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**Lemma 2.** Let \( a_i, b_i \in (0, \infty) \) for \( i = 1, 2, \ldots, n \).

Then,

\[
(a) \quad \sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i b_i \right) \quad \text{if } 1 < p \quad \text{or} \quad p < 0 ;
\]
\[
(b) \quad \sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i b_i \right) \quad \text{if } 0 < p < 1 ;
\]
\[
(c) \quad \text{Jensen’s inequality:}
\]

\[
(a_1^r + a_2^r + \cdots + a_n^r)^\frac{1}{r} < (a_1^s + a_2^s + \cdots + a_n^s)^\frac{1}{s} \quad \text{if } 0 < r < s \quad \text{or} \quad r < s < 0 .
\]
Fuzzy-based Gain Adaptive Scheme for Set-Point Modulated Model Reference Adaptive Controller
www.igi-global.com/article/fuzzy-based-gain-adaptive-scheme-for-set-point-modulated-model-reference-adaptive-controller/217020?camid=4v1a

Mapping with Monocular Vision in Two Dimensions
www.igi-global.com/article/mapping-monocular-vision-two-dimensions/52616?camid=4v1a