A Note on the Uniqueness of Positive Solutions for Singular Boundary Value Problems

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ABSTRACT

In this paper, the authors examine sufficient conditions for the uniqueness of positive solutions of singular Strum-Liouville boundary value problems (BVP) 

\[
\begin{cases}
(E)(\psi(u')u')' + f(t, u, u') = 0, \theta_1 < t < \theta_2 \\
(BC) \left\{ \begin{array}{l}
\alpha_1 u(\theta_1) - \beta_1 u'(\theta_1) = 0 \\
\alpha_2 u(\theta_2) + \beta_2 u'(\theta_2) = 0
\end{array} \right.
\end{cases}
\]

Where \( \alpha_i, \beta_i \geq 0 \) and \( \alpha_i^2 + \beta_i^2 \neq 0 (i = 1, 2) \) are established. The authors use the uniqueness theorems of \((E)\) with respect to the boundary conditions to show that the boundary value problems have one positive solution.

Keywords: Boundary Value Problems, Positive Solutions, Singular, Strum-Liouville, Uniqueness

INTRODUCTION

In this paper, we focus on the uniqueness of positive solutions of boundary value problems for the nonlinear differential equation

\[
(E) \psi(u')u' + f(t, u, u') = 0, \theta_1 < t < \theta_2,
\]

subject to one of the following sets of boundary conditions:

- (BC.1) \( u(\theta_1) = \xi_1 \geq 0, u'(\theta_1) = \xi_2 \geq 0, \)
- (BC.2) \( u'(\theta_1) = \xi_1 \leq 0, u(\theta_2) = \xi_2 \geq 0, \)
- (BC.3) \( u(\theta_1) = \xi_1 \geq 0, u(\theta_2) = \xi_2 \geq 0, \)

where \( (\theta_1, \theta_2) \subseteq (-\infty, \infty), \psi \in C^1([0, \infty)) \) with \( \psi'(y) \gamma \geq 0 \) on \([0, \infty)\) and \( f : (\theta_1, \theta_2) \times (0, \infty) \times (-\infty, 0) \cup (0, \infty) \rightarrow (0, \infty) \)

satisfies:

- (H1) \( f(t, x, y) \) is locally Lipschitz continuous for: 
  \( (x, y) \in (0, \infty) \times \{ (-\infty, 0) \cup (0, \infty) \} \),
- (H2) \( f(t, x, y) / x \) is strictly decreasing with respect to \( x \in (0, \infty) \) for each fixed \((t, y) \in (\theta_1, \theta_2) \times (-\infty, \infty)\),
- (H3) \( \text{sgn}(y) f(t, x, y) \) is decreasing with respect to \( y \in (-\infty, \infty) \) for each fixed \((t, x) \in (\theta_1, \theta_2) \times (0, \infty)\).

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(H4) there is a positive constant $\eta > 0$ such that $f(t,x,y)$ is strictly monotonic with respect to $x \in (0, \eta)$ for each fixed $(t,y) \in (01, 02) \times (-\infty, \infty)$.

Furthermore, we use the uniqueness theorems of $(E)$ with respect to the boundary conditions (BC.i) $(i=1,2,3)$ to show that $(BVP)$ mentioned previously has at most one positive solution in $C^1([\theta_1, \theta_2])$.

Equations of the type $(E)$ arise in studies of radially symmetric solutions (that is, solutions $u$ that depend only on the variable $r = |x|$) of the $m$-Laplacian equation, $\nabla \cdot (|\nabla u|^{m-2} \nabla u) + g(|x|, u, \nabla u) = 0, R_0 < |x| < R_1, x \in \mathbb{R}^n, n \geq 2$.

A radially symmetric solution of the above equation satisfies the ordinary differential equation:

$$\left(|u|^{m-2} u\right)' + \left(\frac{N-1}{r}\right)|u|^{m-2} u' + g(r, u, u') = 0,$$

$R_0 < r < R_1$. With the change of variables, $t = r^{\frac{m-1}{N}}$ (for $m \neq N$ or $t = \log r$ for $m = N$), it can be reduced to an equation of the type:

$$\left(|u|^{m-2} u\right)' + f(t, u, u') = 0, \theta_1 < t < \theta_2, m \geq 3.$$

$$+ \left(\frac{m-1}{m-N}\right)|u|^{m-2} u'' + f(t, u, u') = 0, \theta_1 < t < \theta_2, m \geq 3.$$

Conditions for the existence of solutions of equation $(E)$ with respect to (BC-1)-(BC.3) were studied by many authors: see for instance, De Figueiredo, Lions, and Nussbaum (1982), Granas, Guenther, and Lee (1985), Kaper, Knapp, and Kwong (1991), and Lions (1982). The uniqueness problem concerning $(E)$, for the case $m=2$, has been studied by many authors. For example, Gatica, Oliker, and Waltman (1989), Kwong (1990), Brezis and Oswald (1986) and the excellent book by Agarwal and Lakshmikantham (1993). We further point out that Naito (1995) considered the case $f(t, u, u') = p(t)f(u)$ and established some excellent conditions for uniqueness by using the well-known generalized Prüfer transformation and comparison theorems. In this article, the author attempts to afford a concise approach to study the uniqueness of positive solutions of $(E)$ with boundary conditions (BC.1)-(BC.3) and (BC).

Last in this section, we give the following two notes.

1. It is clear that $(E)$ can be reduced to $(E)$

$$\{\psi(u') + \psi'(u')u\}u'' + f(t, u, u') = 0, \theta_1 < t < \theta_2,$$

(2) The hypothesis $f(t, x, y)$ is Locally Lipschitz continuous for $(x, y) \in (0, \infty) \times \{\mathbb{R} - \{0\}\}$, which guarantees the uniqueness of positive solution of $(E)$ with respect to non-zero initial condition: $u(\eta)u'(\eta) \neq 0$.

**MAIN RESULT**

Let $u$ and $v$ be two distinct positive solution of $(E)$. We define:

$$w(t) = u(t)\psi(v'(t))v'(t) - v(t)\psi(u'(t))u'(t)$$

for $t \in [a,b] \subseteq (\theta_1, \theta_2)$.

It is clear that $w(t)$ satisfies:

$$w'(t) = u(\psi(v')v')' - v(\psi(u')u')'$$

$$+ u'v'\{\psi(v') - \psi(u')\}$$

$$= u\{-f(t,v,v')\} - \{-f(t,u,u')v\}$$

$$+ u'v'\{\psi(v') - \psi(u')\}$$

$$= uv\{\frac{f(t,u,u')}{u} - \frac{f(t,v,v')}{v}\}$$

$$+ u'v'\{\psi(v') - \psi(u')\}$$
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