Chapter 11
Production Planning Models using Max–Plus Algebra

Arun N. Nambiar
California State University, USA

Aleksey Imaev
Ohio University, USA

Robert P. Judd
Ohio University, USA

Hector J. Carlo
University of Puerto Rico - Mayaguez, Puerto Rico

ABSTRACT
The chapter presents a novel building block approach to developing models of manufacturing systems. The approach is based on max-plus algebra. Within this algebra, manufacturing schedules are modeled as a set of coupled linear equations. These equations are solved to find performance metrics such as the make span. The chapter develops a generic modeling block with three inputs and three outputs. It is shown that this structure can model any manufacturing system. It is also shown that the structure is hierarchical, that is, a set of blocks can be reduced to a single block with the same three inputs and three output structure. Basic building blocks, like machining operations, assembly, and buffering are derived. Job shop, flow shop, and cellular system applications are given. Extensions of the theory to buffer allocation and stochastic systems are also outlined. Finally, several numerical examples are given throughout the development of the theory.

INTRODUCTION
Companies around the world are continuously striving to reduce wastes (Womack & Jones, 1996) and improve their operations in an effort to reduce their operating costs. With advances in logistics and distribution, companies are no longer restricted to a geographic region for their market resulting in increased competition. Shrinking product lifecycles, and increasing global competition make it imperative to be able to strike the proverbial iron while it is hot. Companies have to introduce...
new products that cater to the changing needs of consumers as quickly as possible (Anderson, 1997). Lead times assume increased importance and as a result production planning and scheduling becomes critical.

Often times, a company has a plethora of products that it offers to consumers. However, this compounds the scheduling problem as the company needs to determine how much of each product to make and how best to utilize the limited available resources to achieve these production targets. Researchers have developed exact and approximate solutions the various types of scheduling problems commonly encountered in the real world. Exact solutions range from exhaustive enumeration (Morton & Pentico, 1993; Temiz & Erol, 2003) and branch-and-bound techniques (Carlier & Rebai, 1996; Ladhari & Haouari, 2005) to linear programming models (Pinedo, 1995). Approximation methods (Hall, 1998; Sviridenko, 2004) include heuristic approaches such as genetic algorithm (GA) (Goldberg, 1989; Rajendran & Chaudhuri, 1992; Chen et al, 1995; Tang & Liu, 2002; Ravindran et al, 2005), simulated annealing (SA) (Osman & Potts, 1989; Ogbu & Smith, 1990; Ogbu & Smith, 1991; Ishibuchi et al 1995; Chakravarthy & Rajendran, 1999) and more recently ant colony optimization (ACO) (Maniezzo & Carbonaro, 2001; Shyu et al, 2004; Ying & Liao, 2004; Rajendran & Ziegler, 2004; Rajendran & Ziegler, 2005) to name a few. Exact solutions tend to be time-consuming and computationally exhaustive.

This chapter will delve into the basics of max-plus algebra and its properties. It will then explore literature available in the areas of flow-shop scheduling, stochastic scheduling, assembly line balancing, and max-plus algebra. Specific models for flow-shops with and without buffers, batch processing, and assembly lines will be discussed. Finally, the chapter will identify other potential areas where max-plus algebra can be used to develop efficient schedules.

**MAX-PLUS ALGEBRA**

This algebra (The notations used and the concepts given here have been adapted from (Heidergott, 2006)) has two main operators viz. the max operator (maximization) which is denoted by the symbol $\oplus$ and the plus operator (addition) which is denoted by the symbol $\otimes$. The operators are defined as shown in Equations (1) and (2).

$$x \oplus y = \max(x, y) \quad \forall x, y \in \mathbb{R}_{\text{max}}$$  \hspace{1cm} (1)

$$x \otimes y = x + y \quad \forall x, y \in \mathbb{R}_{\text{max}}$$  \hspace{1cm} (2)

where $\mathbb{R}_{\text{max}}$ is the union of the set of real numbers and the zero element of max-plus algebra, $\varepsilon=-\infty$, i.e. $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \varepsilon$. For example

$1 \oplus 2 = \max(1, 2) = 2$

$1 \otimes 2 = 1 + 2 = 3$

The zero element and the unit element in max-plus algebra are $-\infty$ and 0, and are denoted by $\varepsilon$ and $e$, respectively. This is shown in Equations (3) and (4).

$$x \oplus \varepsilon = x \quad \forall x \in \mathbb{R}_{\text{max}}$$  \hspace{1cm} (3)

$$x \otimes e = x \quad \forall x \in \mathbb{R}_{\text{max}}$$  \hspace{1cm} (4)

For example,

$1 \oplus \varepsilon = \max(1, -\infty) = 1$

$1 \otimes e = 1 + 0 = 1$

Further, we also have Equations (5) and (6).

$$x \otimes \varepsilon = \varepsilon \quad \forall x \in \mathbb{R}_{\text{max}}$$  \hspace{1cm} (5)