Approximate Fuzzy Continuity of Functions

Mark Burgin, University of California, USA
Oktay Duman, TOBB Economics and Technology University, Turkey

ABSTRACT
The conventional continuity of a function was further advanced by the concept of approximate continuity introduced by Denjoy to solve some problems of differentiation and integration. According to this new type of continuity, the (classical) continuity conditions may be true not necessarily everywhere, but almost everywhere with respect to some measure, e.g., Borel measure or Lebesgue measure. However, functions that come from real life sources, such as measurement and computation, do not allow, in a general case, to test whether they are continuous or even approximately continuous in the strict mathematical sense. Hence, in this paper, the authors overcome these limitations by introducing and studying the more realistic concept of the approximate fuzzy continuity of functions.

Keywords: Approximate Continuity, Approximate Fuzzy Continuity, Density, Fuzzy Convergence

INTRODUCTION
A natural definition of function continuity is based on the classical concept of sequence convergence. At the same time, there are more general concepts of sequence convergence. One of them is the concept of statistical convergence, introduced by Steinhaus (1951) and Fast (1951) and later reintroduced by Schoenberg (1959). Statistical convergence extends the concept of conventional convergence and is related to many useful constructions in series summability and probability theory.

It seems natural to try to extend the concept of continuity using statistical convergence instead of conventional convergence. However, this brings us back to the same concept of function continuity as the following explanation demonstrates.

Assume that a function \( f: \mathbb{R} \to \mathbb{R} \) is continuous at a point \( a \in \mathbb{R} \) in the conventional sense. Then, it is well-known that \( f(x) \) is defined at \( a \) and for any sequence \( l = \{ a_i \in \mathbb{R}; i = 1, 2, 3, \ldots \} \) that converges to \( a \), the point \( f(a) \) is the limit of the sequence \( \{ f(a_i) \in \mathbb{R}; i = 1, 2, 3, \ldots \} \) or formally, for any sequence \( l = \{ a_i \in \mathbb{R}; i = 1, 2, 3, \ldots \} \), the condition \( \lim_{i \to \infty} a_i = a \) implies \( \lim_{i \to \infty} f(a_i) = f(a) \). Using the statistical continuity instead of conventional continuity, we come to the following definition of statistical continuity. A function \( f: \mathbb{R} \to \mathbb{R} \) is statistically continuous at a point \( a \in \mathbb{R} \) if \( f(x) \) is defined at \( a \) and for any sequence \( l = \{ a_i \in \mathbb{R}; i = 1, 2, 3, \ldots \} \) that converges to \( a \), the point \( f(a) \) is the statistical limit of the sequence \( \{ f(a_i) \in \mathbb{R}; i = 1, 2, 3, \ldots \} \) or formally, for any sequence \( l = \{ a_i \in \mathbb{R}; i = 1, 2, 3, \ldots \} \), the condition \( \lim_{i \to \infty} a_i = a \) implies \( \lim_{i \to \infty} f(a_i) = f(a) \).
\begin{proof}

Recall that a sequence \( l = \{ a_i ; i = 1, 2, 3, \ldots \} \) is statistically convergent to \( a \) or \( a = \text{stat-lim } l \) if \( d(\{ i \in N ; |a_i - a| \geq \varepsilon \} ) = 0 \) for every \( \varepsilon > 0 \) where the asymptotic density \( d(K) \) of a set \( K \subseteq N \) is equal to \( \lim_{n \to \infty} (1/n) |K_n| \) and \( K_n = \{ k \in K ; k \leq n \} \) (Steinhaus, 1951; Fast, 1951; Schoenberg, 1959; Fridy, 1985). Thus, for a statistically continuous at a point \( a \) function \( f \) and any sequence \( l = \{ a_i \in R ; i = 1, 2, 3, \ldots \} \) that converges to \( a \), we have that either the sequence \( \{ f(a_i) \in R ; i = 1, 2, 3, \ldots \} \) converges to \( f(a) \) or there are infinitely many elements \( a_i \) in \( l \) such that \( |f(a_i) - f(a)| \geq \varepsilon \) for some \( \varepsilon \). If the function \( f \) is statistically continuous at the point \( a \) but not continuous at this point, then there is a sequence \( l \) such that \( l \) has many good properties. For example, they have the Darboux property and belong to the first Baire class. Moreover, any bounded approximately continuous function is a derivative of some function.

In this paper, we further advance the concept of approximate continuity in order to study a more general situation where imprecision, vagueness and fuzziness of real processes and systems are taken into account. In the second section, we give some basic notations and properties of approximate continuity and fuzzy continuity, while, in the third section, we introduce and study the concept of approximate fuzzy continuity of a function. In the fourth section, we investigate locally and globally approximately fuzzy continuous functions. The last section of this paper is devoted to concluding remarks.

\section{Density of Sets and Approximate Continuity of Functions}

Let us consider the Lebesgue measure \( m \) on the real line \( R \).

\begin{definition}

The \textit{density} of a measurable subset \( A \) of \( R \) at a point \( a \in R \) is the number

\[ \delta_a \{ A \} = \lim_{\alpha \to 0} \frac{m \{ a - \alpha < x < a + \alpha ; x \in A \} }{2 \alpha} \]

provided the limit exists.

It follows from the definition that if \( \delta_a \{ A \} \) exists, then \( \delta_a \{ A \} = 1 - \delta_a \{ A^c \} \) where \( A^c \) is the complement of \( A \), i.e., \( A^c = R \setminus A \).

\begin{lemma}

For any sets \( A \) and \( B \), if \( \delta_a \{ A \} \) and \( \delta_a \{ B \} \) exist, then we have:

\begin{enumerate}
  \item if \( A \subseteq B \), then \( \delta_a \{ A \} \leq \delta_a \{ B \} \),
  \item \( \delta_a \{ A \cup B \} \leq \delta_a \{ A \} + \delta_a \{ B \} \),
  \item if \( B \subseteq A \), then \( \delta_a \{ A \setminus B \} = \delta_a \{ A \} - \delta_a \{ B \} \),
  \item \( \delta_a \{ A \cap B \} \leq \min \{ \delta_a \{ A \}, \delta_a \{ B \} \} \).
\end{enumerate}
\end{lemma}
\end{proof}

Copyright © 2011, IGI Global. Copying or distributing in print or electronic forms without written permission of IGI Global is prohibited.
The Role of Augmented Reality within Ambient Intelligence
Kevin Curran, Denis McFadden and Ryan Devlin (2013). *Pervasive and Ubiquitous Technology Innovations for Ambient Intelligence Environments* (pp. 73-88).
www.igi-global.com/chapter/role-augmented-reality-within-ambient/68926?camid=4v1a