Chapter 21
Folding Theory for Fantastic Filters in BL–Algebras

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ABSTRACT
In this paper, the author examines the notion of n-fold fantastic and fuzzy n-fold fantastic filters in BL-algebras. Several characterizations of fuzzy n-fold fantastic filters are given. The author shows that every n-fold (fuzzy n-fold) fantastic filter is a filter (fuzzy filter), but the converse is not true. Using a level set of a fuzzy set in a BL-algebra, the author gives a characterization of fuzzy n-fold fantastic filters. Finally, the author establishes the extension property for n-fold and fuzzy n-fold fantastic filters in BL-algebras. The author also constructs some algorithms for folding theory applied to fantastic filters in BL-algebras.

1. INTRODUCTION
Basic logic algebras (BL-algebras for short) introduced by Hájek (1998b) are algebras of Logic BL, their theory is developed in the style of related algebras and logic. The main example of BL-algebras is the unit interval $[0,1]$ endowed with the structure induced by a continuous t-norm. A great deal of literature has been produced on the theory of BCI/BCK/MV/BL-algebras, in particular, emphasis seems to have been put on the ideals and filters theory. From the logical point of view, various ideals and filters correspond to various sets of provable formulas. Zadeh (1965) introduced the notion of fuzzy sets. At present, this concept has been applied to many mathematical branches such as group theory, functional analysis, probability theory, topology and so on. In Lele and Moutari (2008) we have studied the notion of n-fold and fuzzy n-fold various ideals and established many important properties. All the interesting results and the concluding remarks of Jun and Miko (2004) have motivated us to further investigate the foldness of other types of filters in BL-algebras. We find useful to start with the study of foldedness theory of fantastic filters (also called fantastic deductive systems). Thanks to the concept of fuzzy set, we give several characterizations of n-fold
and fuzzy n-fold fantastic filters in BL-algebras. Finally, we give the extension property for n-fold and fuzzy n-fold fantastic filters in BL-algebras. Afterwards, we construct some algorithms to determine whether certain finite structures are BL-algebras, n-fold fantastic filters and fuzzy n-fold fantastic filters. All the above results are the natural generalization of the notion of filters and fuzzy filters (namely deductive and fuzzy deductive systems) in BL-algebras (Lele, 2010, in press; Liu & Li, 2005a, 2005b; Motamed & Saied, 2001; Turunen, Tchikapa, & Lele, in press). It is our hope that this work would serve as a foundation of further study of the theory of some types of (fuzzy) filters in BL-algebra.

2. PRELIMINARIES

A BL-algebra is a structure \((X, \wedge, \vee, *, \rightarrow, 0, 1)\) in which \(X\) is a non-empty set with four binary operations \(\wedge, \vee, *, \rightarrow\) and two constants 0 and 1 satisfying the following axioms:

- **BL-1:** \((X, \wedge, \vee, 0, 1)\) is a bounded lattice;
- **BL-2:** \((X, *, 1)\) is an abelian monoid; which means that \(\ast\) is commutative and associative with \(x \ast 1 = x\);
- **BL-3:** \(x \ast y \leq z\) iff \(x \leq y \rightarrow z\) (residuation);
- **BL-4:** \(x \land y = x \ast (x \rightarrow y)\) (divisibility);
- **BL-5:** \((x \rightarrow y) \lor (y \rightarrow x) = 1\) (prelinearity);

A BL-algebra \(X\) is called a MV-algebra if \(\neg(\neg x) = x\), or equivalently \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\) where \(\neg x = x \rightarrow 0\).

**Definition 2.1.** A t-norm is a binary operation \(T\) over \([0,1]\), that is commutative, associative, monotone, and has 1 as an identity element. \(T\) is a continuous t-norm if it is a t-norm and is a continuous mapping of \([0,1]^2\) into \([0,1]\).

**Example 2.1.** The following are important examples of continuous t-norm:

1. **Lukasiewicz T-Norm:** \(T(x; y) = \max(0; x + y - 1)\).
2. **Gödel T-Norm:** \(T(x; y) = \min(x; y)\).
3. **Product T-Norm:** \(T(x; y) = x \cdot y\)

Note that the dual notion of t-norm is a t-conorm: A t-conorm is a binary operation \(T\) over \([0,1]\), that is commutative, associative, monotone, and has 0 as an identity element.

**Lemma 2.1.** (Hájek, 1998b) Let \(T\) be a continuous t-norm. Then there is a unique operation \(x \rightarrow y\) satisfying, for all \(x, y, z \in [0,1]\), the condition \(T(x, z) \leq y\) iff \(z \leq (x \rightarrow y)\), namely \(x \rightarrow y = \max\{z / T(x, z) \leq y\}\).

**Definition 2.2.** The operation \(x \rightarrow y\) from Lemma 2.1 is called the residuum of the t-norm.

The following operations are residual of the three t-norm:

1. **Lukasiewicz Implication:**
   \[
   x \rightarrow y = \begin{cases} 
   1, & \text{if } x \leq y \\
   \min(1 - x + y; 1), & \text{otherwise}.
   \end{cases}
   \]
2. **Gödel Implication:**
   \[
   x \rightarrow y = \begin{cases} 
   1, & \text{if } x \leq y \\
   y, & \text{otherwise}.
   \end{cases}
   \]
3. **Product Implication:**
   \[
   x \rightarrow y = \begin{cases} 
   1, & \text{if } x \leq y \\
   y / x, & \text{otherwise}.
   \end{cases}
   \]

The main example of BL-algebras is the unit interval \([0; 1]\) endowed with the structure induced by a continuous t-norm. The following properties also hold in any BL-algebra (Agliano & Montagna, 2003) (Cignoli & Torrens, 2005) (Dumitru & Dana, 2003) (Haveshki, Sajed, & Esfami, 2006) (Hájek, 1998b) (Jun, & Miko, 2004) (Turunen, 1999a) (Turunen, 1999b) (Turunen & Sessa, 2001):