Chapter 11
Kolmogorov Superpositions: A New Computational Algorithm

David Sprecher
University of California at Santa Barbara, USA

ABSTRACT

Kolmogorov’s superpositions enable the representation of every real-valued continuous function \( f \) defined on the Euclidean n-cube in the form

\[
f(x, ..., x) = \sum_{j=0}^{2^n} g^j \circ h^j(x, ..., x),
\]

with continuous functions \( g^j \) that compute \( f \), and fixed continuous functions \( h^j = \sum_{p=1}^n \psi^p(x_p) \) dependent only on \( n \). The functions \( h^j \) specify space-filling curves that determine characteristics that are not suitable for efficient computational algorithms. Reversing the process, we specify suitable space-filling curves that enable new functions \( h^j \) that give a computational algorithm better adaptable to applications. Detailed numerical constructions are worked out for the case \( n = 2 \).

INTRODUCTION

This chapter concerns the computation of continuous functions of \( n > 2 \) variables with superpositions of continuous functions of \( m < n \) variables. The function

\[
f(x_1, x_2) = x_1x_2
\]

of two variables can offers a simple example of a representation with superpositions of functions of one variable:

\[
f(x_1, x_2) = g_1 \circ (\psi_{1,1}(x_1) + \psi_{1,2}(x_2)) + g_2 \circ (\psi_{2,1}(x_1) + \psi_{2,2}(x_2)),
\]

where

\[
g_1(t_1) = \frac{1}{4} t_1^2,
\]

\[
g_2(t_2) = -\frac{1}{4} t_2^2,
\]

\[
\psi_{1,1}(x_1) = \psi_{2,1}(x_1) = x_1
\]

and

DOI: 10.4018/978-1-4666-3942-3.ch011
\[ \psi_{1,3}(x_2) = -\psi_{2,3}(x_2) = x_2, \]

i.e.,
\[ x_1 x_2 = \frac{1}{4} (x_1 + x_2)^2 - \frac{1}{4} (x_1 - x_2)^2. \]

In general, we are interested in representations
\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{r} g^i \circ \psi^i(x_1, \ldots, x_n) \]
for some number \( r \) of summands, in which
\[ \psi^i(x_1, \ldots, x_n) \]
are fixed continuous functions that are independent of
\[ f(x_1, \ldots, x_n). \]

In their most general form, such superpositions belong to a class of problems conveniently described by means of a commuting diagram with metric spaces \( X, Y, \) and \( T, \) and a given mapping \( f : X \to Y \) (Figure 1): The problem is to find continuous mappings \( \Psi : X \to T \) and \( g : T \to Y \) such that \( f \) can be replaced with the superposition \( f = g \circ \Psi, \) where \( \Psi \) is a fixed embedding of \( X \) into \( T, \) and when this representation is valid for a sufficiently large class of functions \( f. \) We are not going to pursue here this aspect of generality, and instead will adhere to the setting \( X = E^n, \) in which
\[ E^n = E \times \ldots \times E \]
is the \( n \)-dimensional Euclidean unit cube, \( E = [0,1], \)
\[ T = R^r = \{(t_1, \ldots, t_r)\} \]
is the \( r \)-dimensional Euclidean space, and \( Y = R. \)

What we are leading up to is the remarkable superpositions theorem of Kolmogorov, which states that all real-valued continuous functions defined on \( E^n \) can be represented with superpositions of continuous functions of one variable and addition (Kolmogorov, 1957). Its substance is as follows:

**Theorem 1**

Every real-valued continuous function \( f \) defined on the \( n \)-dimensional Euclidean cube has a representation with sums of monotonic increasing continuous functions \( \psi_p^q \) that are independent of \( f, \) and continuous functions \( g^q \) that compute it:
\[ f(x_1, \ldots, x_n) = \sum_{q=0}^{2^n} g^q \circ \sum_{p=1}^{n} \psi_p^q(x_p) \quad (1) \]

The algorithm that this formula outlines:
\[ (x_1, \ldots, x_n) \to \Sigma \psi_p^q(x_p) \to \Sigma g^q \circ \Sigma_p \psi_p^q(x_p) = f(x_1, \ldots, x_n) \]
is based on a kernel of what might be called arbiter functions,
\[ t^i = \Sigma_p \psi_p^i(x_p); \]