Piecewise Linear Virtual Inputs/Outputs in Interval DEA

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ABSTRACT

A recent development in data envelopment analysis (DEA) concerns the introduction of a piece-wise linear representation of the virtual inputs and/or outputs as a means to model situations where the marginal value of an output (input) is assumed to diminish (increase) as the output (input) increases. Currently, this approach is limited to crisp data sets. In this paper, the authors extend the piece-wise linear approach to interval DEA, i.e., to cases where the input/output data are only known to lie within intervals with given bounds. The authors also define appropriate interval segmentations to implement the piece-wise linear forms in conjunction with the interval bounds of the input/output data and the authors propose a new model compliant with the interval DEA methodology. They finally illustrate their developments with an artificial data set.

Keywords: Crisp Data Sets, Data Envelopment Analysis (DEA), Interval DEA, Marginal Value, Piece-Wise Linear Approach, Piecewise Linear Virtual Inputs/Outputs

INTRODUCTION

Data envelopment analysis (DEA) is the leading technique for assessing the efficiency of decision making units (DMU) in the presence of multiple inputs and outputs. The two milestone DEA models, namely the CCR (Charnes et al., 1978) and the BCC (Banker et al., 1984) models have become standards in the literature of performance measurement. Recent applications of DEA include, among others, those of Mahdavi et al. (2008), Martin and Roman (2010), Pramodth et al. (2008) and Sufian (2010). The underlying mathematical instrument for performing the analysis is linear programming. Performing a typical DEA analysis means solving a series of linear programs, one for each DMU. Efficiency is measured in a bounded ratio scale by the fraction ‘weighted output’ to ‘weighted input’. The inputs and outputs are assumed to be continuous positive variables and the weights are estimated through the associated linear program in favor of the evaluated unit so as to maximize its efficiency.

Focusing on the outputs, an output measure multiplied by the associated weight is called virtual output. The summation of the virtual outputs over all the output dimensions, called

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total virtual output, forms the numerator of the efficiency ratio. A typical interpretation of the weights is that they represent marginal values of outputs. In this manner the virtual outputs can be conceived as linear partial value functions and the total virtual output as an overall additive value function. According to Dyson et al. (2001) the linearity assumption underlying the virtual outputs might be unjustifiable in cases where the marginal value of an output diminishes as the output increases. Recently, Cook and Zhu (2009) and Despotis et al. (2010), motivated by applications involving non-linear virtual outputs, proposed a piece-wise linear representation of the partial value functions as a means to model the situation where particular outputs exhibit diminishing returns. Despotis et al. (2010) showed that ordinary DEA models can be used to perform the efficiency assessments by appropriately introducing additional input/output dimensions in the original data set.

In this paper we extend the piece-wise linear approach in interval DEA to fit the case where the DEA efficiency assessments must be performed on the basis of input and/or output data that are only known to lie within intervals with given bounds (interval data). We reformulate the partial value functions (virtual inputs and outputs) by introducing additional input/output dimensions to obtain an augmented data set that will form the basis for interval efficiency assessments. The rest of the paper unfolds as follows. In the second section we revisit the piece-wise linear DEA models as applied on crisp data to present a simplified formulation, which will be the basis for our new developments. In the third section we provide a brief description of the interval DEA models proposed by Despotis and Smirlis (2002). In the fourth section we provide our main developments that extend the piece-wise linear DEA approach to interval DEA and we formulate appropriate models capable of estimating lower and upper bound efficiencies when inputs (outputs) exhibit increasing (diminishing) returns. In the fifth section we illustrate our new developments with an artificial data set. The paper ends with some concluding remarks.

### DEA MODELS WITH NON-LINEAR PARTIAL VALUE FUNCTIONS

Consider the following input-oriented CCR DEA model (multiplier form) with n DMUs, m inputs and s outputs, where \( y_{ij} \) denotes the level of the output \( r \) \((r=1,\ldots,s)\) produced by the DMU \( j \) \((j=1,\ldots,n)\), \( x_{ij} \) denotes the level of the input \( i \) \((i=1,\ldots,m)\) consumed by the DMU \( j \) and the variables \( u_{rj} \) and \( v_{i} \) are the unknown weights attached to the outputs and the inputs respectively:

\[
\begin{align*}
\text{max} & \quad h_{j} = \sum_{r=1}^{s} u_{rj} y_{rj} \\
\text{s.t.} & \quad \sum_{i=1}^{m} v_{i} x_{ij} = 1 \\
& \quad \sum_{r=1}^{s} u_{rj} - \sum_{i=1}^{m} v_{i} x_{ij} \leq 0, \quad j = 1,\ldots,n \\
& \quad u_{rj}, v_{i} \geq 0 \quad \forall r, i
\end{align*}
\]

Model (1) estimates the relative efficiency \( h_{j} \) of the evaluated DMU \( j \) and is solved repeatedly for every DMU \( j \). Let \( U_{r}(y_{rj}), r = 1,\ldots,s \) and \( U_{i}(x_{ij}), i = 1,\ldots,m \) denote the virtual outputs and inputs for unit \( j \) respectively. Then

\[
U(Y_{j}) = \sum_{r=1}^{s} U_{r}(y_{rj}) = \sum_{r=1}^{s} u_{rj} \quad \text{(3)}
\]

and

\[
U(X_{j}) = \sum_{i=1}^{m} U_{i}(x_{ij}) = \sum_{i=1}^{m} v_{i} x_{ij} \quad \text{(4)}
\]

are the total virtual output and input respectively for unit \( j \), which are linear functions of the weights.

Recently, Cook and Zhu (2009) and Despotis et al. (2010) relaxed the linearity assumption in DEA by introducing a piece-wise linear approach in interval DEA to fit the case where the DEA efficiency assessments must be performed on the basis of input and/or output data that are only known to lie within intervals with given bounds (interval data). We reformulate the partial value functions (virtual inputs and outputs) by introducing additional input/output dimensions to obtain an augmented data set that will form the basis for interval efficiency assessments. The rest of the paper unfolds as follows. In the second section we revisit the piece-wise linear DEA models as applied on crisp data to present a simplified formulation, which will be the basis for our new developments. In the third section we provide a brief description of the interval DEA models proposed by Despotis and Smirlis (2002). In the fourth section we provide our main developments that extend the piece-wise linear DEA approach to interval DEA and we formulate appropriate models capable of estimating lower and upper bound efficiencies when inputs (outputs) exhibit increasing (diminishing) returns. In the fifth section we illustrate our new developments with an artificial data set. The paper ends with some concluding remarks.
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